Contingency Selection in Security Constrained Optimal Power Flow Problem: A Multi-Objective Approach*

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Abstract

We formulate the contingency selection in the security-constrained optimal power flow problem as a multi-objective problem. By taking into account network configuration and transmission line reliability, our approach provides an expanded and improved set of solutions than the conventional N-k criterion. This formulation involves solving mixed-integer nonlinear programs with nonlinear constraints. Algorithms for this class of problems are designed and implemented. Numerical examples including an IEEE 30-bus-41-line network are given to demonstrate the effectiveness of the formulation and the solution technique.

1 Introduction

Electric power is delivered to widely scattered customers through a three-tiered process. It is first produced from a number of different types of generating units of varying capacities and sizes. Transmission networks then carry large amounts of power over a long distance at a high voltage level. From the transmission sources, distribution systems carry the load to a service area by forming a fine network. Under a regulated set-up, these three functions are provided by a given electric utility company which is responsible for supplying power over a specified geographical area and has direct relationships with customers. Under deregulation, which is becoming quite widespread all over the globe, electricity is traded like any other commodity, and the producers and consumers have the option to buy and sell power in a marketplace created to provide competition. Transmission networks can be viewed as consisting of nodes (or buses) and links (or lines). Power is generated and/or consumed at the nodes, and the lines connect these nodes. Electricity has two important characteristics that distinguish it from other commodities. First, it cannot be economically stored. Thus, at every moment, there should be sufficient generation to meet the demand (or load). Second, the amounts of power that flows through the individual transmission lines corresponding to given amounts of injections (i.e., the difference between generation and consumption) at each node cannot be set arbitrarily, but are determined by the laws of physics. The maximum power flow that can be carried out over any one line in a given network is also limited by the physical characteristics of the network, known as the thermal limit.

During the last several years, different market structures have emerged but they all seem to share the feature that the generation and transmission services are unbundled from each other. Under all these schemes the generation services are competitive but the transmission services remain a regulated monopoly that provides open access to the suppliers and consumers of electricity. This latter function is provided by an impartial entity that is known as the

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Independent System Operator (ISO). We quote from Song et al. (2003), "The minimum functions of the ISO should include the operation and coordination of the power system to ensure security, ... the maximum functions of the ISO will include all the reliability-related and market-related functions..."

The term "system security" involves keeping the system operating when some components fail. Even when a transmission network is operating within the physical limits, there always remains the possibility that the individual lines may fail due to accidents, such as a lightning strike, fire, falling trees, harsh weather, or even terrorism. The loss of a single transmission line would change the power flows over the other operating lines, possibly exceeding the physical limits. This might result in cascading failures or even collapse of the entire network. To prevent such a catastrophic result, transmission networks are often operated conservatively so that the system can withstand this kind of contingencies.

With the system security taken into account, a tradeoff between security and economic benefit is inevitable. The more security the system requires, the more are the contingencies that need to be considered, sacrificing more of the economic benefit. Conventionally, the tradeoff is made using the N-k criterion (Ruff 2000), which requires the system to withstand those contingencies in which up to k components have failed. For example, in the network operated by the PJM [1] in the northeastern United States, $k = 1$. In some other instances [2], $k = 2$.

The optimal power flow (OPF) problem (Huneault and Galiana 1991) refers to the nonlinear optimization problem used by the ISO to obtain the optimal amount of the power allocation at various nodes so as to maximize economic benefit without violating any transmission constraint. For different systems and problems, the OPF's could bear different forms of objective functions and constraints. In a security-constrained OPF, system security is enhanced by adding transmission constraints under certain contingency scenarios. Under the N-k criterion, for example, the contingency scenarios are those with at most k components failed.

In reality, however, the N-k criterion may not always provide a good tradeoff between security and benefit. N-1 is sometimes not secure enough, while switching to N-2 or N-3 means too much sacrifice in the benefit. Whereas N-1.5 does not make practical sense, a search for an intermediate tradeoff has been widely considered (Clark 2004, Stott et al. 1987).

We make the point in this paper that the deficiency of the N-k criterion results from its arbitrary choices of contingency scenarios, and more efficient selections of contingencies can be made by taking into account the configuration of the network and the probabilities of individual component failures. To focus on this point, we consider a DC lossless load flow model (Schweppe et al. 1998), in which the transmission constraints reduce to a set of linear (in)equalities. The DC load flow model has been found to be a good approximation to the more accurate AC load flow model, and has been widely used when the thermal limit is the primary concern (Day et al. 2002, Hogan 1993). The contingencies we consider are the failures of transmission lines resulting from uncertainty of the environment.

Various methodologies have been studied to deal with optimization problems involving uncertainties, among which are stochastic programming and robust optimization. (See Birge and Louveaux (1997) and Mulvey et al. (1995) for overviews of stochastic programming and robust optimization, respectively.) We face two issues when trying to formulate this problem as a stochastic programming problem. First, contingency selection is a preventive approach, which means that no second stage decision is considered after the occurrence of contingency. Second, stochastic programming does not generally allow infeasibility under any scenario. Smith et al. (2004) used a penalty function to relax the absolute feasibility requirement, but the penalty is simply assumed to be in proportion to the violation. Infeasibility is usually addressed in robust optimization by chance constraints, in which the choice of infeasibility tolerance needs to be specified exogenously. Mulvey et al. (1995) says that "RO [robust optimization] models are

[1] http://www.pjm.com

[2] House of Commons Trade and Industry Committee. Resilience of the National Electricity Network. Responses to the Committee's Third Report of the Session 2003-04. http://www.publications.parliament.uk/pa/cm200304/cmselect/cmtrdind/630/630.pdf

parametric programs and we have no a priori mechanism for specifying a 'correct' choice of the parameter."

In view of the difficulties arising from the straightforward use of the above approaches, we formulate the contingency selection in the security-constrained OPF problem as a multi-objective optimization problem. Two conflicting objectives — economic benefit and system security — are balanced using a parametric utility function. This approach not only answers the question of which specific contingency scenarios the system should guard against so that the benefit and security are balanced in the most efficient way, but also provides alternative solutions for different security level requirements.

This multi-objective problem involves solving mixed-integer nonlinear programs with nonlinear constraints. Algorithms are designed and implemented to solve this class of problems. Solutions given by this approach are seen to provide more and better tradeoffs than the N-k criterion. We believe that our formulation along with the solution techniques given here provides an additional perspective to deal with uncertainties in optimization problems.

A description of the security-constrained OPF problem is given in Section 2. Section 3 formulates it as a multi-objective optimization problem, and illustrates the advantages of this formulation over the N-k criterion using a 5-bus-6-line example. We present the algorithms for solving this multi-objective problem in Section 4, which is followed by their application to a relatively large system (IEEE 30-bus-41-line network) in Section 5. Section 6 concludes the paper. Appendices are also given to provide more details.

2 Problem Description

This section starts with an introduction and formulation to the OPF problem without security constraints. Then the uncertainty of transmission line failures and how it affects the decision making process are discussed.

2.1 Optimal Power Flow Problem

In an electricity transmission network, a set of nodes $\mathcal{N}$ located at different locations is connected by a set of transmission lines $\mathcal{L}$. The sets of nodes with demand for and supply of power are denoted by $\mathcal{C}$ (consumption) and $\mathcal{P}$ (production), respectively. According to whether there is demand for or supply of power, any node in $\mathcal{N}$ could belong to either $\mathcal{C}$ or $\mathcal{P}$, or both, or neither. Demand and supply functions are assumed to be linear and deterministic (Hobbs et al. 2000):

$$\begin{aligned} q_n^c &\mapsto a_n^c - b_n^c q_n^c,\ \forall n \in \mathcal{C}, \\ q_n^p &\mapsto a_n^p + b_n^p q_n^p,\ \forall n \in \mathcal{P}, \end{aligned}$$

where q_n^c and q_n^p are the quantities (in MWh) of consumption and production of power, respectively. a_n^c, a_n^p (in \$/MWh) and b_n^c, b_n^p (in \$/(MWh)^2) are constant parameters.

The OPF problem we study here relates to that of the ISO who determines how much power is to be consumed and produced at each node at a given hour in order to maximize the social welfare, subject to the transmission constraints (Biggar 2003).

The benefit (social welfare) function $B(\cdot)$ is defined as the consumers' gross surplus less production cost (Chao and Peck 1996):

$$\begin{aligned} B(q) &= \sum_{n\in\mathcal{C}} \int_0^{q_n^c} (a_n^c - b_n^c x)\, dx - \sum_{n\in\mathcal{P}} \int_0^{q_n^p} (a_n^p + b_n^p x)\, dx \\ &= \sum_{n\in\mathcal{C}} \left[a_n^c q_n^c - \frac{1}{2} b_n^c \left(q_n^c\right)^2 \right] - \sum_{n\in\mathcal{P}} \left[a_n^p q_n^p + \frac{1}{2} b_n^p \left(q_n^p\right)^2 \right], \end{aligned}$$

where $q = \begin{bmatrix} q^c \\ q^p \end{bmatrix}$.

If a DC lossless network is assumed, then the transmission constraints can be linearized, which include thermal and balancing constraints. Each transmission line l has a thermal limit T_l, which is the maximum amount of power it can carry (in either direction) without causing heat-related deterioration. The thermal constraints require that the power flow through each transmission line should not exceed its thermal limit. The balancing constraint means that, with power loss through transmission ignored, total power production should equal total consumption.

Unlike regular networks, power flows through the electricity network always obey Kirchhoff's laws (Biggar 2003): (i) For any node, the total amount of power flows going into and out of the node are equal; and (ii) For any closed loop, the sum of directed power flows around the loop is zero. Therefore, after ISO makes the decision about power production and consumption at each node, the power flow through the entire network will be automatically determined by Kirchhoff's laws. A Power Transfer Distribution Factors (PTDF) matrix (Biggar 2003) is used to calculate the power flow through each transmission line from the power production and consumption at each node. In a DC network, the PTDF matrix is of size $|\mathcal{L}| \times |\mathcal{N}|$ (with one redundant column), and is exclusively determined by the topology of the network and the physical characteristics of the transmission lines (resistance and reactance) (Schweppe et al. 1998). Let $H \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{N}|}$ denote the PTDF matrix, then the power flow through line l is

$$\sum_{n \in \mathcal{C}} H_{l,n} q_n^p - \sum_{n \in \mathcal{P}} H_{l,n} q_n^c.$$

Then the thermal constraints can be expressed as

$$\left| \sum_{n \in \mathcal{C}} H_{l,n} q_n^p - \sum_{n \in \mathcal{P}} H_{l,n} q_n^c \right| \leq T_l, \ \forall l \in \mathcal{L}.$$

The balancing constraint is

$$\sum_{n \in \mathcal{C}} q_n^c - \sum_{n \in \mathcal{P}} q_n^p = 0.$$

(The details about the calculation of PTDF matrix and transmission constraint in a DC load flow model are given in Appendix A.)

Therefore, the ISO's OPF problem is:

$$\max_q \quad B(q) = \sum_{n \in \mathcal{C}} \left[a_n^c q_n^c - \frac{1}{2} b_n^c \left(q_n^c\right)^2 \right] - \sum_{n \in \mathcal{P}} \left[a_n^p q_n^p + \frac{1}{2} b_n^p \left(q_n^p\right)^2 \right] \tag{1}$$

$$\text{s. t.} \quad \left| \sum_{n \in \mathcal{C}} H_{l,n} q_n^p - \sum_{n \in \mathcal{P}} H_{l,n} q_n^c \right| \leq T_l, \ \forall l \in \mathcal{L} \tag{2}$$

$$\sum_{n \in \mathcal{C}} q_n^c - \sum_{n \in \mathcal{P}} q_n^p = 0 \tag{3}$$

$$q_n^c \geq 0, \ \forall n \in \mathcal{C}; \ 0 \leq q_n^p \leq \overline{q_n^p}, \ \forall n \in \mathcal{P}, \tag{4}$$

where $\overline{q_n^p}$ is the capacity of the generators at node n. Coefficients a_n^c, b_n^c, a_n^p, b_n^p and $\overline{q_n^p}$ are assumed to be non-negative and appropriately valued, so that the problem possesses an optimal solution.

By rewriting constraint (2) as

$$\sum_{n \in \mathcal{C}} H_{l,n} q_n^p - \sum_{n \in \mathcal{P}} H_{l,n} q_n^c \leq T_l, \ \forall l \in \mathcal{L}$$
$$\sum_{n \in \mathcal{C}} H_{l,n} q_n^p - \sum_{n \in \mathcal{P}} H_{l,n} q_n^c \geq -T_l, \ \forall l \in \mathcal{L},$$

we can transform the OPF into a quadratic program with linear constraints, the matrix form of which will be used throughout the remainder of this paper:

$$\begin{aligned} \max_{q} \quad & B(q) = c^\top q + \frac{1}{2} q^\top Q q \\ \text{s.t.} \quad & Aq \leq b \\ & q \in \mathbb{R}_+^m, \end{aligned}$$

where $m = |\mathcal{C}| + |\mathcal{P}|$, and vectors and matrices c, Q, A, and b are appropriately valued. Q is a diagonal matrix with non-positive components, so this problem is a concave quadratic maximization program.

2.2 Security-Constrained OPF

System security is the ability to withstand contingencies, in other words, to remain intact even after outages or equipment failures occur (Shahidehpour et al. 2005). Security plays a crucial role in the operation of a power system, but absolute system security can never be achieved.

There are two principal classes of methods to enhance the system security: *preventive* and *corrective* methods. The former enables the system to be prepared for certain contingencies beforehand, while the latter helps make quick and correct responses to the contingencies after their occurrence. In this paper, we focus on the preventive method. To facilitate the analysis, we make the following definitions and assumptions.

Transmission line failure

Each transmission line l can be in either of two states: working or failed. We define a binary random variable Y_l which assumes the value 1 when it is working and 0 otherwise. When a line has failed, we assume it is no longer able to carry any power flow, and is physically off the network.

Contingency scenarios: $\mathcal{S}$

A contingency scenario s is defined as a binary vector $(y_1^s, ..., y_{|\mathcal{L}|}^s) \in \mathbb{B}^{|\mathcal{L}|}$ indicating the states of transmission lines. Denote by $\mathcal{S}$ the complete set of scenarios. We have $|\mathcal{S}| = 2^{|\mathcal{L}|}$.

Probability of scenario: p_s

For any scenario $s \in \mathcal{S}$ and its corresponding transmission line state vector $(y_1^s, ..., y_{|\mathcal{L}|}^s) \in \mathbb{B}^{|\mathcal{L}|}$, the probability of scenario s is

$$p_s := P\left[Y_1 = y_1^s, ..., Y_{|\mathcal{L}|} = y_{|\mathcal{L}|}^s\right].$$

Appendix B provides a framework for computing these probabilities. Here we assume that they are known constants. While a scenario with $p_s = 0$ theoretically exists, one could safely eliminate it from $\mathcal{S}$. We assume throughout that $p_s > 0, \forall s \in \mathcal{S}$. We also have $\sum_{s \in \mathcal{S}} p_s = 1$.

Transmission constraint: $A^s q \leq b^s$

Every contingency scenario has its own network topology, and thus a unique PTDF matrix. Therefore, the transmission constraints differ under different scenarios, even in dimension. For any $s \in \mathcal{S}$, we denote the set of constraints under scenario s by $A^s q \leq b^s$. Also denote by r_s the number of rows of A^s.

Scenario partition: $\mathcal{S}^0(q)$ and $\mathcal{S}^1(q)$

For any fixed decision $q \in \mathbb{R}^m_+$, we partition the complete set of scenarios $\mathcal{S}$ into two subsets:

$$\mathcal{S}^0(q) := \{s \in \mathcal{S} : A^s q > b^s\},$$

and

$$\mathcal{S}^1(q) := \{s \in \mathcal{S} : A^s q \le b^s\}.$$

That is, $\mathcal{S}^0(q)$ and $\mathcal{S}^1(q)$ are the subsets of scenarios under which the decision q is infeasible and feasible, respectively.

System infeasibility

After a decision q is made, some transmission lines may suffer an outage. When this happens, the topology of the network is changed, and power flow through the entire network will then be automatically re-routed according to the new network topology. Let $\hat{s}$ denote the scenario that corresponds to the new topology. If $A^{\hat{s}} q > b^{\hat{s}}$, or in words, if the re-routed power flow violates the transmission constraints, then a system infeasibility occurs.

Infeasibility cost: C_s

In the multi-objective approach being proposed, we need an estimate of the cost associated with system infeasibility, because it will affect decision making. Intuitively, the higher the infeasibility cost, the more secure we would want the power system to be.

Under decision q and an infeasible scenario $s \in \mathcal{S}^0(q)$, the benefit is assumed to be $[B(q) - C_s]$, where C_s is the infeasibility cost. The value of C_s is indeed random, because the consequences of an overload, which indicates system infeasibility, vary from happy endings to catastrophes, depending on the responses made afterwards. If the overload is quickly detected and all failed transmission lines are repaired and restored to the network in time, the system simply gets back on track, in which case $C_s \approx 0$; on the other hand, if overload on one line is left unattended for a long time, it could burn down and cause cascading failures of other lines, and eventually lead to a system collapse or blackout, where C_s could be in the order of millions of dollars. For simplicity, we assume that $C_s > 0$ is a known constant, representing an average value of infeasibility cost under scenario s.

N-k criterion

In Section 2.1, the OPF problem assumes the best scenario (call it s^0), under which all transmission lines are working, thus the PTDF matrix is derived according to the full network topology. The optimal solution q^* to (1), however, is only guaranteed to be "optimal" under s^0. Should any other scenario occur involving transmission line failure(s), q^* might not even be feasible. To enhance system security and reduce the risk of system infeasibility, additional contingency scenarios should be taken into account in the OPF problem. Nevertheless, considering all possible scenarios to ensure absolute security is not a sensible proposition. As a matter of fact, under the worst scenario, in which all transmission lines have failed, there is no feasible solution excepting when the entire transmission system is left unused thus not creating any economic benefit.

Typically, the tradeoff between economic benefit and system security is made by considering a pre-determined subset of scenarios using the N-k criterion, which requires the system to withstand up to k out of N component failures.

Under the N-k criterion, the security-constrained OPF problem becomes

$$\begin{aligned} \max_{q} \quad & B(q) = c^\top q + \frac{1}{2} q^\top Q q \\ \text{s.t.} \quad & A^s q \le b^s, \ \forall s \in \mathcal{S}^{\text{N-k}} \\ & q \in \mathbb{R}^m_+, \end{aligned}$$

where $\mathcal{S}^{\text{N-k}}$ is the subset of scenarios with at most k transmission lines failed:

$$\mathcal{S}^{\text{N-k}} := \left\{ s \in \mathcal{S} : \sum_{l \in \mathcal{L}} y_l^s \geq |\mathcal{L}| - k \right\}.$$

While this criterion can effectively enhance system security by preventing system infeasibility under certain contingencies, it has the following disadvantages:

1. Important information about the transmission line reliability and network topology is disregarded. Thus, this criterion may be ignoring failures of some unreliable and critical transmission lines, while considering possible failures of reliable and uncritical ones;
2. The difference between N-k and N-(k+1) may be significant, which means that there are few or no intermediate choices between a secure but unbeneficial decision and a insecure but beneficial one;
3. The cost of system infeasibility is not explicitly considered; and
4. Sometimes there may exist solutions that are both more beneficial and more secure than those given by the N-k criterion.

3 Multi-Objective Formulation

In this section, we present a multi-objective formulation that will

1. Quantitatively take into account the probabilities of transmission line failures and topology of the network, so that only the failures of the most unreliable and/or critical transmission lines are considered;
2. Provide an expanded and improved set of solutions to the decision maker compared to those of the N-k criterion;
3. Explicitly consider the tradeoff among benefit, infeasibility cost and risk of infeasibility; and
4. Guarantee the optimality of solutions under a probabilistic criterion.

3.1 Formulations MOF1(α) and MOF2(α)

For any $s \in \mathcal{S}$, a binary decision variable x_s is introduced to indicate whether the contingency scenario s should be considered to ensure the feasibility of q ($x_s = 1$) or not ($x_s = 0$). The contingency selection is then determined by the following optimization program:

$$\begin{aligned} \text{MOF1}(\alpha): \qquad \max_{x,q} \quad & \alpha B(q) + \sum_{s \in \mathcal{S}} C_s p_s x_s \\ \text{s. t.} \quad & A^s q \leq b^s, \quad \text{if } x_s = 1, \ \forall s \in \mathcal{S} \\ & A^s q > b^s, \quad \text{if } x_s = 0, \ \forall s \in \mathcal{S} \\ & x \in \mathbb{B}^{|\mathcal{S}|}, q \in \mathbb{R}_+^m. \end{aligned} \tag{5}$$

To explain the motivation behind this formulation, we first make the following definitions.

Definition. *Define the* feasibility function $f(\cdot) : \mathbb{B}^{|\mathcal{S}|} \mapsto [0, 1]$ *as*

$$f(x) := \sum_{s \in \mathcal{S}} p_s x_s.$$

For a given feasible solution (x, q) to MOF1(α), $f(x)$ evaluates the system security in terms of the probability that q is feasible. If p_s is perceived as the expected proportion of time that the system operates under scenario s, then $f(x)$ gives the expected proportion of time that q is feasible.

Definition. *Define the* prevention function $g(\cdot) : \mathbb{B}^{|\mathcal{S}|} \mapsto [0,1]$ *as*

$$g(x) := \frac{\sum_{s\in\mathcal{S}} C_s p_s x_s}{\sum_{s\in\mathcal{S}} C_s p_s}.$$

Prevention function is a cost weighted feasibility function. For a given feasible solution (x, q) to MOF1(α), $g(x)$ evaluates the system security in terms of the percentage of infeasibility cost that is prevented by (x, q).

Definition. *For any* $x \in \mathbb{B}^{|\mathcal{S}|}$, $q \in \mathbb{R}^m_+$, *and* $\alpha \in [0, \infty)$, *define the* utility function *as*

$$U_\alpha(x, q) := \alpha B(q) + \sum_{s\in\mathcal{S}} C_s p_s \cdot g(x) = \alpha B(q) + \sum_{s\in\mathcal{S}} C_s p_s x_s.$$

Consider a feasible solution (x, q) to MOF1(α). Under a feasible scenario $s \in \mathcal{S}^1(q)$, the benefit is $B(q)$; under an infeasible scenario $s \in \mathcal{S}^0(q)$, the benefit is $B(q) - C_s$. If the ISO is a risk-neutral decision maker, then his objective would be to maximize the expected benefit

$$\begin{aligned} & \sum_{s\in\mathcal{S}^1(q)} B(q)p_s + \sum_{s\in\mathcal{S}^0(q)} [B(q) - C_s]p_s \\ = \quad & B(q) - \sum_{s\in\mathcal{S}^0(q)} C_s p_s \\ = \quad & B(q) - \sum_{s\in\mathcal{S}} C_s p_s (1 - x_s) \\ = \quad & B(q) - \left(\sum_{s\in\mathcal{S}} C_s p_s - \sum_{s\in\mathcal{S}} C_s p_s x_s \right), \end{aligned}$$

which is equivalent to maximizing the utility function $U_\alpha(x, q)$ with $\alpha = 1$. Notice that $B(q)$ is the definite benefit that one gets for certain, while $\left(\sum_{s\in\mathcal{S}} C_s p_s - \sum_{s\in\mathcal{S}} C_s p_s x_s\right)$ depends on the uncertain cost C_s that is incurred randomly. Therefore, we introduce the parameter α to represent risk preferences of different decision makers over "hard" benefit and "soft" cost. Risk-averse and risk-loving decision makers would prefer to maximize the utility function $U_\alpha(\cdot, \cdot)$ with $0 \le \alpha < 1$ and $\alpha > 1$, respectively. By maximizing the utility function with a decision-maker-determined α, one gets the most efficient balance of benefit and risk. The optimal solution x^* indicates the contingency selection decision, while q^* gives the optimal power allocation in light of the selected contingencies.

The optimal solutions to the utility functions with different α could be different. We define a solution q to be *utility-optimal* if there exists an $\alpha \in [0, +\infty)$ such that q is an optimal solution to MOF1(α). In contrast, a solution is said to be *dominated* if it is not utility-optimal, because for any $\alpha \in [0, +\infty)$, there exists another solution that dominates it in the utility function. For a specific power system at a given time, the ISO would have a fixed risk preference value of α, thus the utility-optimal solution to MOF1(α) gives the best tradeoff for him. But the ISO may not be aware of his value of α exactly, we thus set our goal to find the complete set of utility-optimal solutions, and the range of α for each solution to remain optimum. A set of utility-optimal solutions is said to be *complete* if for any $\alpha \in [0, +\infty)$, there exists a solution in the set that is optimal to MOF1(α). The complete set of utility-optimal solutions connected by segments of utility functions will be referred to as the *utility frontier*. The ISO can thus choose his favorite tradeoff from the utility frontier without exactly knowing his α beforehand, and he also has the option of choosing different tradeoffs for different system requirements.

By relaxing the strict inequality (5), we get

$$\begin{aligned} \text{MOF2}(\alpha): \qquad & \max_{x,q} \quad \alpha B(q) + \sum_{s\in\mathcal{S}} C_s p_s x_s \\ & \text{s.t.} \quad A^s q \le b^s, \quad \text{if } x_s = 1, \ \forall s \in \mathcal{S} \\ & \qquad\quad x \in \mathbb{B}^{|\mathcal{S}|}, q \in \mathbb{R}^m_+. \end{aligned}$$

Table 1: Node data of the 5-bus-6-line example

node	demand function	supply function
A	N/A	$q \mapsto 7 + 0.0452q, \quad 0 \le q \le 210$
B	$q \mapsto 100 - 0.2629q$	N/A
C	$q \mapsto 100 - 0.2550q$	$q \mapsto 23.51 + 0.0600q, \quad 0 \le q \le 520$
D	$q \mapsto 100 - 0.2333q$	$q \mapsto 15 + 0.1210q, \quad 0 \le q \le 200$
E	N/A	$q \mapsto 5 + 0.0092q, \quad 0 \le q \le 600$

The following proposition proves the equivalence of MOF1(α) and MOF2(α) under optimum.

Proposition 1. *(x, q) is an optimal solution to MOF1(α) if and only if (x, q) is an optimal solution to MOF2(α).*

Proof. Let (x^2, q^2) be any optimal solution to MOF2(α). Then it suffices to prove that for all $s \in \mathcal{S}$, $A^s q^2 \le b^s \Leftrightarrow x_s^2 = 1$ (and thus $A^s q^2 > b^s \Leftrightarrow x_s^2 = 0$).

$\Leftarrow$: (x^2, q^2) is a feasible solution to MOF2(α).

$\Rightarrow$: Prove by contradiction. Suppose there exists a scenario $s' \in \mathcal{S}$ such that $A^{s'} q^2 \le b^{s'}$ and $x_{s'} = 0$. Define $\hat{x} \in \mathbb{B}^{|\mathcal{S}|}$ as $\hat{x}_s = \begin{cases} 1, & \text{if } A^s q^2 \le b^s; \\ 0, & \text{otherwise.} \end{cases}$ $, \forall s \in \mathcal{S}$, then $(\hat{x}, q^2)$ is a feasible solution to MOF2(α). Moreover, $U_\alpha(\hat{x}, q^2) - U_\alpha(x^2, q^2) \ge C_{s'} p_{s'} x_{s'} > 0$, which contradicts the assumption that (x^2, q^2) is an optimal solution to MOF2(α). $\square$

3.2 Multi-Objective Formulation vs. the N-k Criterion

When the contingency selection x is pre-determined as

$$x_s = \begin{cases} 1, & s \in \mathcal{S}^{\text{N-k}}, \\ 0, & \text{otherwise,} \end{cases}$$

MOF2(α) reduces to the security-constrained OPF problem under the N-k criterion, and it is independent of α. Therefore, the N-k solutions are special feasible solutions to MOF2(α), but not necessarily the optimal ones.

The difference between utility-optimal solutions and N-k solutions can be demonstrated with the following example. Figure 1 is a 5-bus-6-line network example from PJM website [1], in which $\mathcal{N}$ ={A, B, C, D, E}, $\mathcal{C}$ ={B, C, D}, $\mathcal{P}$ ={A, C, D, E}, $\mathcal{L}$ ={A-B, B-C, C-D, D-E, E-A, A-D}. Details about the network parameters are given in Tables 1 and 2. There are $2^6 = 64$ scenarios in $\mathcal{S}$. The probability of failure of a transmission line l, $P(Y_l = 0)$, is randomly generated from a uniform distribution U(0,0.04). For computational simplicity, we assume that Y_l's are independent, so that the probability of a scenario s can be conveniently obtained from

$$p_s = P\left(Y_1 = y_1^s, ..., Y_{|\mathcal{L}|} = y_{|\mathcal{L}|}^s\right) = \prod_{l \in \mathcal{L}} P(Y_l = y_l^s).$$

Infeasibility cost C_s is assumed to be

$$C_s = 10000\left(5 - \frac{\sum_{l \in \mathcal{L}} y_l^s}{|\mathcal{L}|}\right), \forall s \in \mathcal{S}.$$

This example is solved using both the N-k criterion and the multi-objective approach. The algorithms that we use to obtain the utility-optimal solutions will be given in Section 4.

The N-k and utility-optimal solutions are listed in Tables 3 and 4, respectively, and illustrated in Figure 2. We can see that N-0, N-2 and N-3 coincide with utility-optimal solutions 1, 8 and 9, respectively, but N-1 is dominated. Utility-optimal solutions intermediate between N-0 and

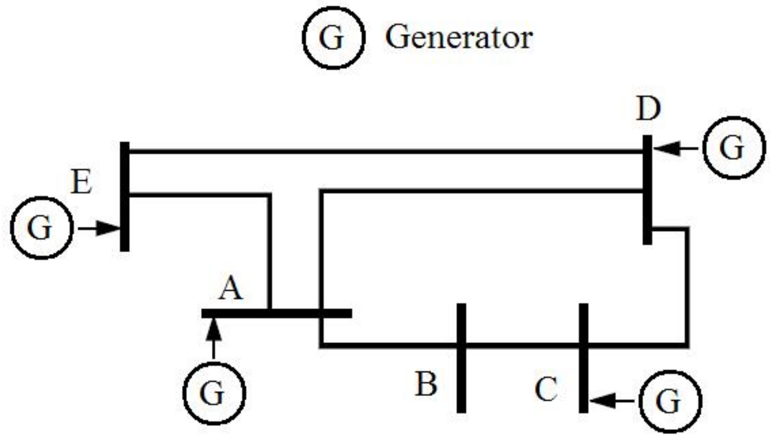

Figure 1: A 5-bus-6-line network example

Table 2: Transmission line data of the 5-bus-6-line example

line number	line	resistance R (Ω)	reactance X (Ω)	thermal limit (MWh)	probability of failure
1	A-B	0	0.0281	377	0.2494%
2	B-C	0	0.0108	77	3.8443%
3	C-D	0	0.0297	223	1.9658%
4	D-E	0	0.0297	240	2.5701%
5	E-A	0	0.0064	360	0.5435%
6	A-D	0	0.0304	159	1.1895%

Table 3: N-k solutions of the 5-bus-6-line example

Solution	$f(x)$	$g(x)$	$B(q)$
N-0	0.90032	0.68397	46,826
N-1	0.99762	0.99216	31,591
N-2	0.99998	0.99992	21,964
N-3	1.00000	1.00000	19,202

Table 4: Utility-optimal solutions of the 5-bus-6-line example

Solution number	$f(x)$	$g(x)$	$B(q)$	Range of α within which solution is optimal
1	0.90032	0.68397	46,826	$0.8185 \leq \alpha$
2	0.95437	0.85506	44,074	$0.2978 \leq \alpha \leq 0.8185$
3	0.96586	0.89151	42,463	$0.2623 \leq \alpha \leq 0.2978$
4	0.99615	0.98760	37,640	$0.0186 \leq \alpha \leq 0.2623$
5	0.99735	0.99156	34,832	$0.0184 \leq \alpha \leq 0.0186$
6	0.99932	0.99778	30,384	$0.0080 \leq \alpha \leq 0.0184$
7	0.99978	0.99926	27,944	$0.0015 \leq \alpha \leq 0.0080$
8	0.99998	0.99992	21,964	$0.0004 \leq \alpha \leq 0.0015$
9	1.00000	1.00000	19,202	$0 \leq \alpha \leq 0.0004$

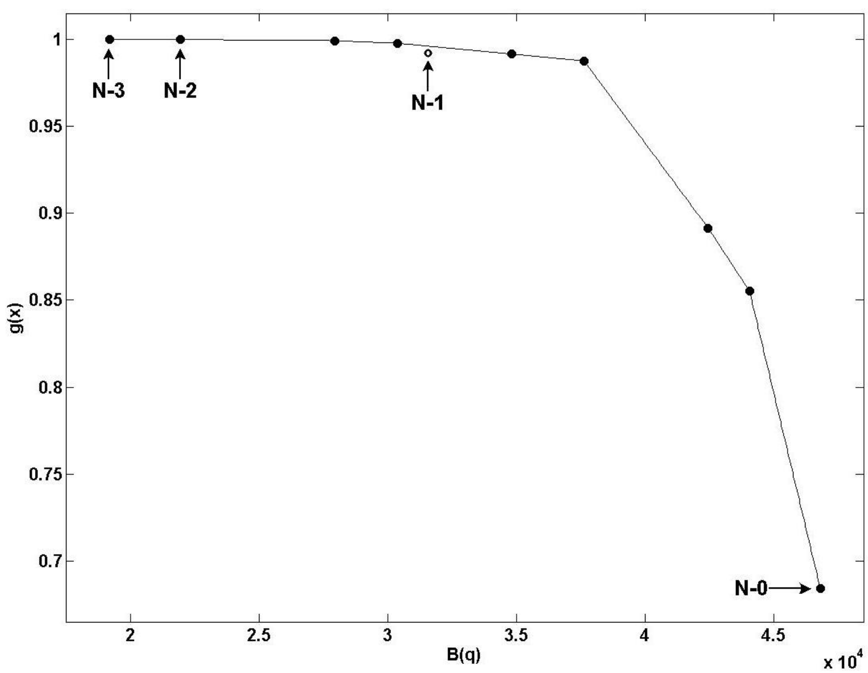

Figure 2: Utility frontier of the 5-bus-6-line example

N-1 and between N-1 and N-2 are also provided by this multi-objective approach. Connected by segments of utility functions, the utility-optimal solutions are "optimal" in the sense that no feasible solution exists beyond the frontier (in the increasing directions of both axes), while all feasible solutions within the frontier are dominated.

The difference between $f(x)$ and $g(x)$ results from the different perspectives of risk evaluation used in their definitions. Taking the first solution (x^1, q^1) as an example, $f(x^1) = 0.90032$ means that in about 90% of the time, q^1 is feasible; while $g(x^1) = 0.68397$ means that (x^1, q^1) prevents about 68% of all expected infeasibility cost. The more C_s differ among scenarios, the more $f(x)$ and $g(x)$ differ from each other. If C_s is independent of s, $f(x)$ and $g(x)$ become the same.

4 Solution Techniques

In this section, we give two algorithms. Algorithm 1 is for solving MOF2(α) for a specific value of α, and algorithm 2 is for finding the complete set of optimal solutions on the utility frontier. Effectiveness of these algorithms for larger systems will be demonstrated in Section 5 with an IEEE 30-bus-41-line network example, in which a set of 862 scenarios is considered.

4.1 Preliminaries

Consider the following problem:

$$\begin{aligned}
\text{MOF3}(\alpha,\, d): \qquad & \max_{x,q} \quad \alpha B(q) + \sum_{s \in \mathcal{S}} C_s p_s \cdot g(x) \\
& \text{s.t.} \quad A^s q \le b^s + (1 - x_s) d^s, \ \forall s \in \mathcal{S} \\
& \qquad\quad x \in \mathbb{B}^{|\mathcal{S}|}, q \in \mathbb{R}^m_+,
\end{aligned}$$

where, for all $s \in \mathcal{S}$, d^s is a parametric vector with the same dimension as b^s, and $d = \begin{bmatrix} d^1 \\ \vdots \\ d^{|\mathcal{S}|} \end{bmatrix}$.

Let (x^2, q^2) be an optimal solution to MOF2(α). Denote the set of optimal solutions to MOF2(α) by

$$\begin{aligned}
\mathcal{XQ}^2(\alpha) := \{(x, q) : & \alpha B(q) + \sum_{s \in \mathcal{S}} C_s p_s \cdot g(x) = \alpha B(q^2) + \sum_{s \in \mathcal{S}} C_s p_s \cdot g(x^2), \\
& A^s q \le b^s \text{ if } x_s = 1, \ \forall s \in \mathcal{S}, \\
& x \in \mathbb{B}^{|\mathcal{S}|}, q \in \mathbb{R}^m_+\}.
\end{aligned}$$

Proposition 2. *For any $\alpha \in [0, +\infty)$, there exists a finite vector $d(\alpha)$ such that the following inequality holds for all $(x, q) \in \mathcal{XQ}^2(\alpha)$:*

$$A^s q \le b^s + (1 - x_s) d^s(\alpha), \ \forall s \in \mathcal{S}.$$

Proof. Consider any $\alpha \in [0, +\infty)$. Let (x^0, q^0) be any feasible solution to MOF2(α). Then the following inequality is satisfied for all $(x^2, q^2) \in \mathcal{XQ}^2(\alpha)$:

$$\alpha B(q^2) + \sum_{s \in \mathcal{S}} C_s p_s \cdot g(x^2) \ge \alpha B(q^0) + \sum_{s \in \mathcal{S}} C_s p_s \cdot g(x^0),$$

then

$$\alpha B(q^2) \ge \alpha B(q^0) + \sum_{s \in \mathcal{S}} C_s p_s \cdot [g(x^0) - g(x^2)] \ge \alpha B(q^0) + \sum_{s \in \mathcal{S}} C_s p_s [g(x^0) - 1],$$

or equivalently

$$c^\top q^2 + \frac{1}{2}(q^2)^\top Q q^2 \geq B(q^0) + \frac{1}{\alpha}\sum_{s\in\mathcal{S}} C_s p_s[g(x^0) - 1].$$

Now consider the following problem

$$\begin{aligned} \max_{q} \quad & a_i^s q \\ \text{s.t.} \quad & c^\top q + \frac{1}{2} q^\top Q q \geq B(q^0) + \frac{1}{\alpha}\sum_{s\in\mathcal{S}} C_s p_s[g(x^0) - 1] \\ & q \in \mathbb{R}_+^m, \end{aligned} \tag{6}$$

where a_i^s is the i^{th} row of A^s. This problem is a linear program with quadratic constraints, whose feasible region is a convex compact set, and thus has a finite optimal solution. Denote by $q_i^s(\alpha)$ the optimal solution to (6) corresponding to a_i^s. For all $s \in \mathcal{S}$ and $i \in \{1, ..., r_s\}$, define

$$\tilde{d}_i^s(\alpha) := \max\{a_i^s q_i^s(\alpha) - b_i^s, 0\}, \tag{7}$$

then for all $(x, q) \in \mathcal{XQ}^2(\alpha)$, $\tilde{d}^s(\alpha)$ satisfies

$$A^s q \leq b^s + (1 - x_s)\tilde{d}^s(\alpha), \ \forall s \in \mathcal{S}.$$

□

Proposition 3. *For any $\alpha \in [0, +\infty)$, $(x, q) \in \mathcal{XQ}^2(\alpha)$, and $s \in \mathcal{S}$, let $d^s(\alpha)$ satisfy*

$$A^s q \leq b^s + (1 - x_s)d^s(\alpha).$$

Then $(x, q) \in \mathcal{XQ}^2(\alpha)$ if and only if (x, q) is an optimal solution to MOF3(α, d).

Proof. $\Rightarrow$: By the definition of $d(\alpha)$, for any $(x^2, q^2) \in \mathcal{XQ}^2(\alpha)$, (x^2, q^2) is feasible to MOF3(α, d). (x^2, q^2) is also an optimal solution to MOF3(α, d), because any feasible solution to MOF3(α, d) is also feasible to MOF2(α, d).

$\Leftarrow$: For any optimal solution (x^3, q^3) to MOF3(α, d), we know (x^3, q^3) is feasible to MOF2(α). By the ($\Rightarrow$) part, we have

$$\alpha B(q^3) + \sum_{s\in\mathcal{S}} C_s p_s \cdot g(x^3) = \alpha B(q^2) + \sum_{s\in\mathcal{S}} C_s p_s \cdot g(x^2),$$

therefore, (x^3, q^3) is also an optimal solution to MOF2(α). □

MOF3(α, d) has only linear constraints, and is thus more tractable than MOF2(α). By proposition 3, we can solve MOF3(α, d) for a sufficiently large d to get the optimal solution to MOF2(α). Proposition 2 guarantees the existence of a finite d, and the proof shows how to obtain it.

4.2 Algorithm 1

Algorithm 1 is to find an optimal solution to MOF3(α, d) for a specific value of $\alpha \in [0, +\infty)$.

Step 0. Select an error tolerance parameter $\delta \geq 0$.

Step 1. Solve for (η^*, x^*) to the master problem

$$\text{(PM):} \qquad \begin{aligned} \max_{\eta, x} \quad & \alpha\eta + \sum_{s\in\mathcal{S}} C_s p_s x_s \\ \text{s.t.} \quad & \text{cuts, if any} \\ & \eta \in \mathbb{R}_+, x \in \mathbb{B}^{|\mathcal{S}|}. \end{aligned}$$

If (PM) is infeasible, then stop, and MOF3(α, d) is infeasible. Otherwise continue to step 2.

Step 2. Solve for (q^*, λ^*) to the subproblem

$$\begin{aligned} \max_{q} \quad & c^\top q + \frac{1}{2} q^\top Q q \\ \text{s.t.} \quad & A^s q \le b^s + (1 - x_s^*) d^s \ \ : (\lambda^s), \quad \forall s \in \mathcal{S} \\ & q \in \mathbb{R}_+^m, \end{aligned} \tag{8}$$

where, for all $s \in \mathcal{S}$, $\lambda^s \ge 0$ is the dual variable for its corresponding constraint. x^* is the optimal solution obtained in step 1.
If $\eta^* \le c^\top q^* + \frac{1}{2} (q^*)^\top Q q^* + \delta \eta^*$, then stop; otherwise add the following cut to master problem (PM) and go back to step 1:

$$\begin{aligned} & \eta + \left[\left(d^1\right)^\top \left(\lambda^1\right)^*, ..., \left(d^{|\mathcal{S}|}\right)^\top \left(\lambda^{|\mathcal{S}|}\right)^* \right] x \\ \le \quad & c^\top q^* + \frac{1}{2} (q^*)^\top Q q^* + \left[\left(d^1\right)^\top \left(\lambda^1\right)^*, ..., \left(d^{|\mathcal{S}|}\right)^\top \left(\lambda^{|\mathcal{S}|}\right)^* \right] x^*. \end{aligned} \tag{9}$$

Theorem 1. *If the number of scenarios is finite, algorithm 1 finitely converges to an optimal solution when it exists or proves the infeasibility of MOF3(α, d).*

Proof. Notice that MOF3(α, d) is equivalent to the following problem

$$\text{MOF4}(\alpha,\ d): \qquad \begin{aligned} \max_{\eta, x} \quad & \alpha \eta + \sum_{s \in \mathcal{S}} C_s p_s x_s \\ \text{s.t.} \quad & \eta \le \mathcal{B}(x) \\ & \eta \in \mathbb{R}_+, x \in \mathbb{B}^{|\mathcal{S}|}, \end{aligned}$$

where

$$\mathcal{B}(x) = \max_{q \in \mathbb{R}_+^m} \left\{ c^\top q + \frac{1}{2} q^\top Q q : A^s q \le b^s + (1 - x_s) d^s,\ \forall s \in \mathcal{S} \right\}.$$

Now consider the following problem:

$$\beta(x) = \min_{\lambda \in \mathbb{R}_+^{\sum_{s \in \mathcal{S}} r^s}, p \in \mathbb{R}^m} \left\{ \sum_{s \in \mathcal{S}} \left[b^s + (1 - x_s) d^s \right]^\top \lambda^s - \frac{1}{2} p^\top Q p : \sum_{s \in \mathcal{S}} \left(A^s\right)^\top \lambda^s - Q p \ge c \right\}.$$

By comparing the KKT conditions of $\mathcal{B}(x)$ and $\beta(x)$, one can see that for any $x^0 \in \mathbb{B}^{|\mathcal{S}|}$, if q^0 is the optimal solution to $\mathcal{B}(x)$, then there exists an optimal solution (λ^0, p^0) to $\beta(x)$ with $p^0 = q^0$, and $\mathcal{B}(x^0) = \beta(x^0)$. Let (η^0, x^0) be any feasible solution to MOF4(α, d) and (λ^0, p^0) be the optimal solution to $\beta(x^0)$. The following (in)equalities hold for any feasible solution (η, x) to MOF4(α, d):

$$\begin{aligned} \eta & \le \mathcal{B}(x) \\ & = \beta(x) \\ & \le \sum_{s \in \mathcal{S}} \left[b^s + (1 - x_s) d^s \right]^\top \left(\lambda^s\right)^0 - \frac{1}{2} \left(p^0\right)^\top Q p^0 \\ & = \sum_{s \in \mathcal{S}} \left[b^s + \left(1 - (x_s)^0\right) d^s \right]^\top \left(\lambda^s\right)^0 - \frac{1}{2} \left(p^0\right)^\top Q p^0 + \sum_{s \in \mathcal{S}} \left[(x_s)^0 - x_s \right]^\top \left(\lambda^s\right)^0 \\ & = \beta(x^0) + \sum_{s \in \mathcal{S}} \left[(x_s)^0 - x_s \right]^\top \left(\lambda^s\right)^0 \\ & = \mathcal{B}(x^0) + \sum_{s \in \mathcal{S}} \left[(x_s)^0 - x_s \right]^\top \left(\lambda^s\right)^0 \\ & = c^\top q^0 + \frac{1}{2} \left(q^0\right)^\top Q q^0 + \left[\left(d^1\right)^\top \left(\lambda^1\right)^0, ..., \left(d^{|\mathcal{S}|}\right)^\top \left(\lambda^{|\mathcal{S}|}\right)^0 \right] (x^0 - x), \end{aligned}$$

which justify constraint (9).

The finite convergence follows from the fact that there is a finite number of feasible solutions x. $\square$

4.3 Algorithm 2

Algorithm 2 obtains all the optimal solutions on the utility frontier as the value of α varies within $[0, +\infty)$.

Step 1. Solve for q^* to the following problem

$$\begin{aligned} \max_{q} \quad & c^\top q + \frac{1}{2} q^\top Q q \\ \text{s.t.} \quad & A^s q \le b^s, \quad \forall s \in \mathcal{S} \\ & q \in \mathbb{R}_+^m. \end{aligned}$$

Set $\underline{B} = c^\top q^* + \frac{1}{2} (q^*)^\top Q q^*$ and $\overline{g} = 1$.

Step 2. Set $\overline{B} = \max \left\{ c^\top q + \frac{1}{2} q^\top Q q : q \in \mathbb{R}_+^m \right\}$ and $\underline{g} = \sum_{s \in \mathcal{S}^1(q^*)} C_s p_s / \sum_{s \in \mathcal{S}} C_s p_s$.

Step 3. Define $\mathcal{B} = \left\{ \left(\underline{B}, \overline{B} \right) \right\}$, $\mathcal{G} = \left\{ \left(\overline{g}, \underline{g} \right) \right\}$, and $\mathcal{U} = \left\{ \left(\underline{B}, \overline{g} \right), \left(\overline{B}, \underline{g} \right) \right\}$.

Step 4. If $\mathcal{B} \neq \emptyset$ and $\mathcal{G} \neq \emptyset$, then

1. Let (B_1, B_2) and (g_1, g_2) be the first elements in $\mathcal{B}$ and $\mathcal{G}$, respectively;
2. Calculate $\alpha = \sum_{s \in \mathcal{S}} C_s p_s (g_2 - g_1) / (B_1 - B_2)$, and determine a vector d accordingly using (7);
3. Solve for an optimal solution (B_α^*, g_α^*) to MOF3(α, d) using the algorithm in Section 4.2;
4. If $(B_\alpha^*, g_\alpha^*) \neq (B_1, g_1)$ and $(B_\alpha^*, g_\alpha^*) \neq (B_2, g_2)$, then $\mathcal{B} = \mathcal{B} \cup \{ (B_\alpha^*, B_1), (B_\alpha^*, B_2) \}$, $\mathcal{G} = \mathcal{G} \cup \{ (g_\alpha^*, g_1), (g_\alpha^*, g_2) \}$, and $\mathcal{U} = \mathcal{U} \cup (B_\alpha^*, g_\alpha^*)$;
5. $\mathcal{B} = \mathcal{B} \backslash (B_1, B_2)$, $\mathcal{G} = \mathcal{G} \backslash (g_1, g_2)$; and
6. Repeat step 4.

Otherwise stop, and $\mathcal{U}$ is the complete set of optimal solutions on the utility frontier.

Theorem 2. *Algorithm 2 obtains a complete set of utility-optimal solutions.*

Proof. First, algorithm 2 terminates finitely, because there is a finite number of feasible solutions x. Suppose it terminates with N utility-optimal solutions: $\mathcal{U} = \{(B_1, g_1), ..., (B_N, g_N)\}$ with $B_1 < \cdots < B_N$ and $g_1 > \cdots > g_N$, where B and g represent the values of $B(q^*)$ and $g(x^*)$ in MOF3(α, d), respectively. For all $i, j \in \{1, ..., N\}$, define $\alpha_{i,j} = \sum_{s \in \mathcal{S}} C_s p_s (g_j - g_i)/(B_i - B_j)$. We know that $\alpha_{i-1,i} < \alpha_{i,i+1}, \forall i = 2, ..., N-1$, because of utility-optimality of $\mathcal{U}$.

Step 1 obtains the most conservative solution $(\underline{B}, \overline{g}) = (B_1, g_1)$, which is optimal to MOF3(α, d) for $0 \le \alpha \le \alpha_{1,2}$. Step 2 obtains the least conservative solution $(\overline{B}, \underline{g}) = (B_N, g_N)$, which is optimal to MOF3(α, d) for $\alpha \ge \alpha_{N-1,N}$. We see from step 4(4) that, for all $i = 2, ..., N-1$, (B_i, g_i) is optimal to MOF3(α, d) for both $\alpha = \alpha_{i-1,i}$ and $\alpha = \alpha_{i,i+1}$.

Finally, show that for $i = 2, ..., N-1$, (B_i, g_i) is optimal to MOF3(α, d) for any $\alpha \in (\alpha_{i-1,i}, \alpha_{i,i+1})$. As a matter of fact, for any $(\hat{B}, \hat{g}) \in \mathbb{R}_+ \times [0, 1]$ and $\hat{\alpha} \in (\alpha_{i-1,i}, \alpha_{i,i+1})$, we have

$$
\begin{aligned}
& \left(\hat{\alpha} B_i + \sum_{s\in\mathcal{S}} C_s p_s g_i\right) - \left(\hat{\alpha}\hat{B} + \sum_{s\in\mathcal{S}} C_s p_s \hat{g}\right) \\
= \; & \frac{\alpha_{i,i+1} - \hat{\alpha}}{\alpha_{i,i+1} - \alpha_{i-1,i}} \left[\left(\alpha_{i-1,i} B_i + \sum_{s\in\mathcal{S}} C_s p_s g_i\right) - \left(\alpha_{i-1,i}\hat{B} + \sum_{s\in\mathcal{S}} C_s p_s \hat{g}\right)\right] \\
& + \frac{\hat{\alpha} - \alpha_{i-1,i}}{\alpha_{i,i+1} - \alpha_{i-1,i}} \left[\left(\alpha_{i,i+1} B_i + \sum_{s\in\mathcal{S}} C_s p_s g_i\right) - \left(\alpha_{i,i+1}\hat{B} + \sum_{s\in\mathcal{S}} C_s p_s \hat{g}\right)\right] \\
\geq \; & 0.
\end{aligned}
$$

□

5 An IEEE 30-Bus-41-Line Network Example

The algorithms presented in Section 4 are implemented with MATLAB 7.0.4 [3] and CPLEX 9.0 [4], and are applied to the IEEE 30-bus-41-line example [5]. The configuration of the network is shown in Figure 3, and detailed node and transmission line data are given in Tables 5 and 6. The data are taken from Alsac and Stott (1973) and Hobbs et al. (2000). Probabilities of transmission line failures are randomly generated from a uniform distribution U(0,0.015). Infeasibility cost C_s is assumed to be

$$C_s = 1000\left(5 - \frac{\sum_{l\in\mathcal{L}} y_l^s}{|\mathcal{L}|}\right), \forall s \in \mathcal{S}.$$

For a network of this size, a complete implementation of the algorithms is virtually impossible. As a matter of fact, in this example, $|\mathcal{L}| = 41$, $|\mathcal{S}| = 2^{|\mathcal{L}|} = 2^{41} \approx 2 \times 10^{12}$. Therefore, in problem (PM), the number of variables is $2^{41} + 1$.

Instead of considering a complete set of scenarios $\mathcal{S}$, here we consider a small subset of scenarios $\mathcal{S}'$ where at most two transmission lines are failed. Then $|\mathcal{S}'| = \sum_{i=0}^{2} \left(\begin{array}{c} 41 \\ i \end{array}\right) = 862$, and $\sum_{s\in\mathcal{S}'} p_s = 0.99741$. The prevention function is calculated within the subset of $\mathcal{S}'$, so in this example

$$g(x) = \frac{\sum_{s\in\mathcal{S}'} C_s p_s x_s}{\sum_{s\in\mathcal{S}'} C_s p_s},$$

and it gives the percentage of expected infeasibility cost under contingencies $\mathcal{S}'$ that can be prevented.

In both this example and the smaller one in Section 3.2, we have restricted ourselves to the utility-optimal solutions that are feasible under at least one scenario. This is done by changing $\overline{B}$ in algorithm 2 to

$$\overline{B} = \max\left\{c^\top q + \frac{1}{2} q^\top Q q : A^{s^0} q \leq b^{s^0}, q \in \mathbb{R}_+^m\right\}.$$

By doing so, we eliminate the utility-optimal solution that is infeasible under all scenarios (even the best one s^0). While this solution is mathematically "optimal" for an extremely risk-loving decision maker, it is practically unthinkable.

The N-k and utility-optimal solutions are listed in Tables 7 and 8, respectively, and illustrated in Figure 4. N-0, N-1 and N-2 solutions coincide with utility-optimal solutions 1, 4 and 7,

[3] `http://www.mathworks.com`
[4] `http://www.ilog.com`
[5] `http://www.ee.washington.edu/research/pstca/pf30/pg_tca30bus.htm`

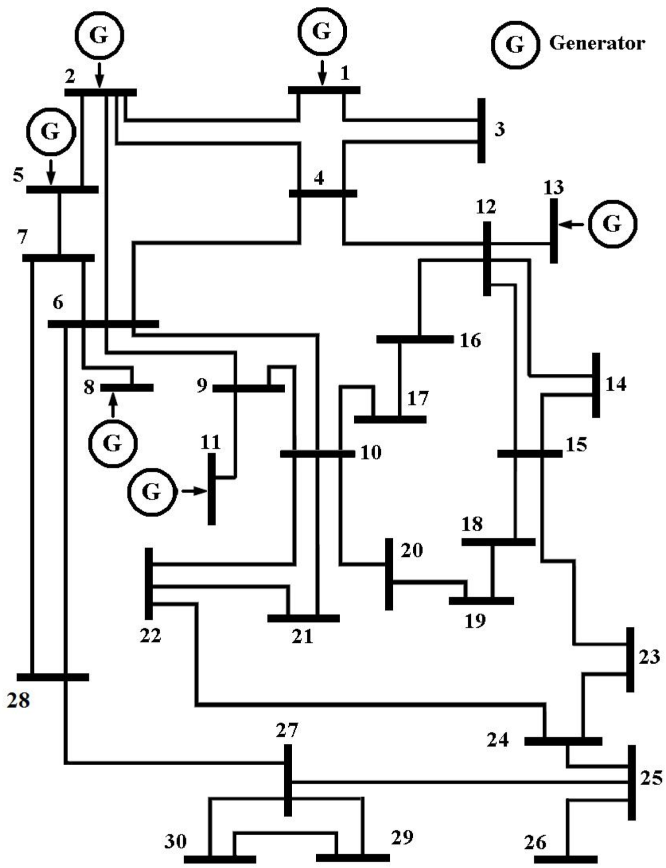

Figure 3: IEEE 30-bus-41-line network

Table 5: Node data of the 30-bus-41-line example

node	demand function	supply function	
1	N/A	$q \mapsto 0 + 2.00q,$	$0 \le q \le 200$
2	$q \mapsto 40 - 0.2304q$	$q \mapsto 0 + 1.75q,$	$0 \le q \le 80$
3	$q \mapsto 40 - 2.0833q$	N/A	
4	$q \mapsto 40 - 0.6579q$	N/A	
5	$q \mapsto 40 - 0.0531q$	$q \mapsto 0 + 1.00q,$	$0 \le q \le 50$
6	N/A	N/A	
7	$q \mapsto 40 - 0.2193q$	N/A	
8	$q \mapsto 40 - 0.1667q$	$q \mapsto 0 + 3.25q,$	$0 \le q \le 35$
9	N/A	N/A	
10	$q \mapsto 40 - 0.8621q$	N/A	
11	N/A	$q \mapsto 0 + 3.00q,$	$0 \le q \le 30$
12	$q \mapsto 40 - 0.4464q$	N/A	
13	N/A	$q \mapsto 0 + 3.00q,$	$0 \le q \le 40$
14	$q \mapsto 40 - 0.8065q$	N/A	
15	$q \mapsto 40 - 0.6098q$	N/A	
16	$q \mapsto 40 - 1.4286q$	N/A	
17	$q \mapsto 40 - 0.5556q$	N/A	
18	$q \mapsto 40 - 1.5625q$	N/A	
19	$q \mapsto 40 - 0.5263q$	N/A	
20	$q \mapsto 40 - 2.2727q$	N/A	
21	$q \mapsto 40 - 0.2857q$	N/A	
22	N/A	N/A	
23	$q \mapsto 40 - 1.5625q$	N/A	
24	$q \mapsto 40 - 0.5747q$	N/A	
25	N/A	N/A	
26	$q \mapsto 40 - 1.4286q$	N/A	
27	N/A	N/A	
28	N/A	N/A	
29	$q \mapsto 40 - 2.0833q$	N/A	
30	$q \mapsto 40 - 0.4717q$	N/A	

Table 6: Transmission line data of the 30-bus-41-line example

line number	line	resistance R (Ω)	reactance X (Ω)	thermal limit (MWh)	probability of failure
1	1-2	0.0192	0.0575	130	1.4705%
2	1-3	0.0452	0.1852	130	0.4781%
3	2-4	0.0570	0.1737	65	0.9308%
4	3-4	0.0132	0.0379	130	0.2523%
5	2-5	0.0472	0.1983	130	0.7458%
6	2-6	0.0581	0.1763	65	0.4358%
7	4-6	0.0119	0.0414	90	0.8567%
8	5-7	0.0460	0.1160	70	1.0431%
9	6-7	0.0267	0.0820	130	1.2155%
10	6-8	0.0120	0.0420	32	1.2099%
11	6-9	0	0.2080	65	0.4767%
12	6-10	0	0.5560	32	1.0459%
13	9-11	0	0.2080	65	0.6875%
14	9-10	0	0.1100	65	1.2737%
15	4-12	0	0.2560	65	0.4532%
16	12-13	0	0.1400	65	0.9324%
17	12-14	0.1231	0.2559	32	0.2100%
18	12-15	0.0662	0.1304	32	0.2195%
19	12-16	0.0945	0.1987	32	0.6097%
20	14-15	0.2210	0.1997	16	0.7552%
21	16-17	0.0824	0.1932	16	0.1503%
22	15-18	0.1070	0.2185	16	0.2676%
23	18-19	0.0639	0.1292	16	0.5326%
24	19-20	0.0340	0.0680	32	0.2730%
25	10-20	0.0936	0.2090	32	0.5097%
26	10-17	0.0324	0.0845	32	0.9870%
27	10-21	0.0348	0.0749	32	1.0654%
28	10-22	0.0727	0.1499	32	0.9882%
29	21-22	0.0116	0.0236	32	0.6989%
30	15-23	0.1000	0.2020	16	0.4093%
31	22-24	0.1150	0.1790	16	1.0361%
32	23-24	0.1320	0.2700	16	0.2423%
33	24-25	0.1885	0.3292	16	0.6479%
34	25-26	0.2544	0.3800	16	0.9444%
35	25-27	0.1093	0.2087	16	0.4459%
36	28-27	0	0.3960	65	0.6801%
37	27-29	0.2198	0.4153	16	0.8327%
38	27-30	0.3202	0.6027	16	0.4581%
39	29-30	0.2399	0.4533	16	0.5680%
40	8-28	0.0636	0.2000	32	0.3078%
41	6-28	0.0169	0.0599	32	0.0647%

Table 7: N-k solutions of the 30-bus-41-line example

Solution	$f(x)$	$g(x)$	$B(q)$
N-0	0.97190	0.94017	2,312
N-1	0.99670	0.99834	1,826
N-2	0.99741	1.00000	1,431

Table 8: Utility-optimal solutions of the 30-bus-41-line example

Solution number	$f(x)$	$g(x)$	$B(q)$	Range of α within which solution is optimal
1	0.97190	0.94017	2,312	$24.7918 \leq \alpha$
2	0.98098	0.96145	2,311	$0.1503 \leq \alpha \leq 24.7918$
3	0.98998	0.98258	2,070	$0.1109 \leq \alpha \leq 0.1503$
4	0.99670	0.99834	1,826	$0.0939 \leq \alpha \leq 0.1109$
5	0.99714	0.99937	1,807	$0.0238 \leq \alpha \leq 0.0939$
6	0.99728	0.99969	1,784	$0.0015 \leq \alpha \leq 0.0238$
7	0.99741	1.00000	1,431	$0 \leq \alpha \leq 0.0015$

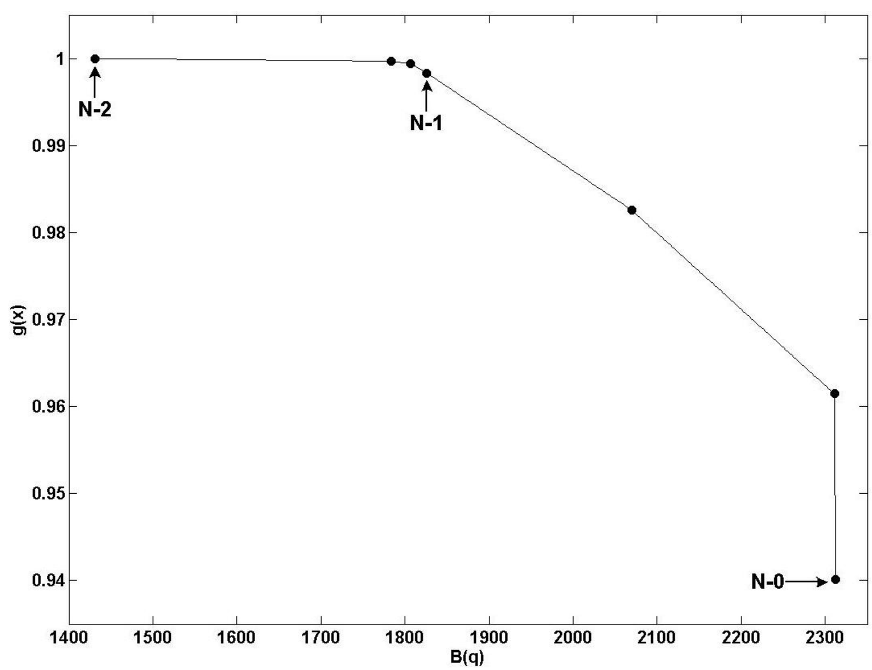

Figure 4: Utility frontier of the 30-bus-41-line example

respectively. For those ISO's who plan to increase the system security from N-1 to a higher level but are reluctant to use the N-2 criterion because of the marked drop in benefit, the utility frontier provides two alternative solutions 5 and 6, which are more secure than N-1, and are much more economically beneficial than N-2.

6 Conclusion

This paper has presented a multi-objective approach of contingency selection in the security-constrained optimal power flow problem. This approach is able to provide an expanded and improved set of tradeoffs between economic benefit and system security than the conventional N-k criterion. The multi-objective formulation quantitatively evaluates the influence of each contingency by taking into account the probabilities of transmission failures and network topology, and it selects only the most critical contingencies to consider so that benefit and security are balanced efficiently.

This approach also presents a new framework to deal with uncertainties in optimization problems. It allows infeasibility under rare scenarios, and searches for different optimal tradeoffs according to risk preferences. The algorithms given in this paper can also be applied to problems outside the power industry that fit in the multi-objective framework.

One difficulty with this approach is the size of the networks that can be handled. For a network with $|\mathcal{L}|$ transmission lines, the master problem (PM) will have $2^{|\mathcal{L}|}+1$ variables, thus it is virtually impossible for the entire algorithms to be applied to large networks. However, as the example in Section 5 has demonstrated, even if applied to only a small subset of scenarios $\mathcal{S}'$ with $\sum_{s\in\mathcal{S}'} p_s$ close to 1, this approach still dominates the N-k criterion.

Appendix

A PTDF Matrix and Transmission Constraint Calculation

Once the net injection (power generation less consumption) into each node is determined, the power flow through the electric network will be uniquely and automatically determined by the laws of physics. The PTDF matrix H is used to calculate the power flow through each transmission line. Given the generation and consumption at each node, the power flow through line l can be calculated as

$$\sum_{n\in\mathcal{N}} H_{l,n}(q_n^p - q_n^c) = \sum_{n\in\mathcal{C}} H_{l,n}q_n^p - \sum_{n\in\mathcal{P}} H_{l,n}q_n^c.$$

In the DC load flow model, the PTDF matrix has a closed form expression (Schweppe et al. 1998):

$$H_{|\mathcal{L}|\times(|\mathcal{N}|-1)} = \Omega A(A^\top \Omega A)^{-1},$$

where $\Omega \in \mathbb{R}^{|\mathcal{L}|\times|\mathcal{L}|}$ is the reactance-resistance matrix, and $A \in \mathbb{R}^{|\mathcal{L}|\times(|\mathcal{N}|-1)}$ is the reduced sized arc-node incidence matrix. The original arc-node incidence matrix is a $|\mathcal{L}|$ by $|\mathcal{N}|$ matrix with a column rank of $|\mathcal{N}|-1$, thus an arbitrary column needs to be deleted from A, otherwise $(A^\top\Omega A)$ will be a singular matrix. The PTDF matrix we get from the above formula has one less column than $|\mathcal{N}|$, to which we add a zero column to retrieve a size of $|\mathcal{L}|$ by $|\mathcal{N}|$.

The set of transmission constraints includes:

$$\sum_{n\in\mathcal{C}} H_{l,n}q_n^p - \sum_{n\in\mathcal{P}} H_{l,n}q_n^c \le T_l,\ \forall l\in\mathcal{L}$$
$$\sum_{n\in\mathcal{C}} H_{l,n}q_n^p - \sum_{n\in\mathcal{P}} H_{l,n}q_n^c \ge -T_l,\ \forall l\in\mathcal{L}$$
$$\sum_{n\in\mathcal{C}} q_n^c - \sum_{n\in\mathcal{P}} q_n^p = 0$$
$$q_n^p \le \overline{q_n^p},\ \forall n\in\mathcal{P}.$$

Take the 5-bus-6-line network as an example, in which

$$\Omega = \operatorname{diag}\left[\frac{X_l}{R_l^2+X_l^2}\right] = \begin{bmatrix} 35.5872 & 0 & 0 & 0 & 0 & 0 \\ 0 & 92.5926 & 0 & 0 & 0 & 0 \\ 0 & 0 & 33.6700 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.6700 & 0 & 0 \\ 0 & 0 & 0 & 0 & 156.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 32.8947 \end{bmatrix}$$

and

$$A_{|\mathcal{L}|\times|\mathcal{N}|} = \begin{array}{c} \\ A-B \\ B-C \\ C-D \\ D-E \\ E-A \\ A-D \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}.$$

If we choose the column corresponding to bus E (this bus is called swing bus) in $A_{|\mathcal{L}|\times|\mathcal{N}|}$ as the redundant column, then the PTDF matrix under the best scenarios s^0 is:

$$H^{s^0}_{|\mathcal{L}|\times|\mathcal{N}|} = \begin{array}{c} \\ A-B \\ B-C \\ C-D \\ D-E \\ E-A \\ A-D \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0.0344 & -0.6354 & -0.5085 & -0.1595 & 0 \\ 0.0344 & 0.3646 & -0.5085 & -0.1595 & 0 \\ 0.0344 & 0.3646 & 0.4915 & -0.1595 & 0 \\ 0.1120 & 0.2629 & 0.3209 & 0.4805 & 0 \\ -0.8880 & -0.7371 & -0.6791 & -0.5195 & 0 \\ 0.0776 & -0.1017 & -0.1706 & -0.3600 & 0 \end{bmatrix} \end{array},$$

and then the transmission constraint $A^{s^0} q \leq b^{s^0}$ is

$$\begin{bmatrix} -0.6354 & -0.5085 & -0.1595 & -0.0344 & 0.5085 & 0.1595 & 0 \\ 0.3646 & -0.5085 & -0.1595 & -0.0344 & 0.5085 & 0.1595 & 0 \\ 0.3646 & 0.4915 & -0.1595 & -0.0344 & -0.4915 & 0.1595 & 0 \\ 0.2629 & 0.3209 & 0.4805 & -0.1120 & -0.3209 & -0.4805 & 0 \\ -0.7371 & -0.6791 & -0.5195 & 0.8880 & 0.6791 & 0.5195 & 0 \\ -0.1017 & -0.1706 & -0.3600 & -0.0776 & 0.1706 & 0.3600 & 0 \\ 0.6354 & 0.5085 & 0.1595 & 0.0344 & -0.5085 & -0.1595 & 0 \\ -0.3646 & 0.5085 & 0.1595 & 0.0344 & -0.5085 & -0.1595 & 0 \\ -0.3646 & -0.4915 & 0.1595 & 0.0344 & 0.4915 & -0.1595 & 0 \\ -0.2629 & -0.3209 & -0.4805 & 0.1120 & 0.3209 & 0.4805 & 0 \\ 0.7371 & 0.6791 & 0.5195 & -0.8880 & -0.6791 & -0.5195 & 0 \\ 0.1017 & 0.1706 & 0.3600 & 0.0776 & -0.1706 & -0.3600 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q^c_B \\ q^c_C \\ q^c_D \\ q^p_A \\ q^p_C \\ q^p_D \\ q^p_E \end{bmatrix} \leq \begin{bmatrix} 377 \\ 77 \\ 223 \\ 240 \\ 360 \\ 159 \\ 377 \\ 77 \\ 223 \\ 240 \\ 360 \\ 159 \\ 0 \\ 0 \\ 210 \\ 520 \\ 200 \\ 600 \end{bmatrix}.$$

To illustrate how the constraints differ under different scenarios, let us consider another scenario s^1, in which the two most unreliable lines B-C and D-E are failed. Under scenario s^1, the arc-node incidence matrix is

$$A^{s^1}_{|\mathcal{L}|\times|\mathcal{N}|} = \begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A-B \\ C-D \\ E-A \\ A-D \end{matrix} & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array},$$

the PTDF matrix is

$$H^{s^1}_{|\mathcal{L}|\times|\mathcal{N}|} = \begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A-B \\ C-D \\ E-A \\ A-D \end{matrix} & \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array},$$

and the transmission constraint $A^{s^1} q \leq b^{s^1}$ is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q^c_B \\ q^c_C \\ q^c_D \\ q^p_A \\ q^p_C \\ q^p_D \\ q^p_E \end{bmatrix} \leq \begin{bmatrix} 377 \\ 223 \\ 360 \\ 159 \\ 377 \\ 223 \\ 360 \\ 159 \\ 0 \\ 0 \\ 210 \\ 520 \\ 200 \\ 600 \end{bmatrix}.$$

B Estimation of p_s

Computationally, the probability of scenario p_s can be obtained using the historical data of transmission line states. We may consider a historical period of time $[0, T]$ for which transmission reliability data is available. The states of transmission line l during this period can be regarded as a continuous time binary stochastic process $Y_l(t)$. The probability of scenario s can be estimated by the average expected proportion of the time during which the transmission lines are in state $(y_1^s, ..., y_{|\mathcal{L}|}^s) \in \mathbb{B}^{|\mathcal{L}|}$:

$$p_s := \frac{1}{T}\int_0^T P\left[Y_1(t) = y_1^s, ..., Y_{|\mathcal{L}|}(t) = y_{|\mathcal{L}|}^s \middle| Y_1(0), ..., Y_{|\mathcal{L}|}(0)\right] dt.$$

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