Identification of Harmonic Sources by Underdetermined State Estimator

Huaiwei Liao
Department of Electrical and Computer Engineering
Carnegie Mellon University
Pittsburgh, PA 15213
hliao@andrew.cmu.edu

***Abstract*— This paper presents a new system-wide harmonic state estimation method with the capability to identify harmonic sources with fewer meters than state variables. Note there are only a few simultaneous harmonic sources among the suspicious buses. By extending the concept of observability, the underdetermined system can be observable when considering the sparsity of harmonic sources. We formulate harmonic state estimation as a constrained sparsity maximization problem. It can be solved by linear programming. Our numerical experiments in IEEE 14-bus power systems show the effectiveness of the proposed method.**

I. Introduction

Harmonic pollution is recognized as an important factor in the degradation of power quality, which may shorten equipment life and interfere with communication and control devices [1]. In consequence, the IEEE Harmonic Standard [2] recommends requirements for utilities and customers to minimize the harmonic contents in power networks. To effectively alleviate harmonic pollution, it is important to identify harmonic sources and estimate the distribution of harmonic voltages and currents by real-time measurements.

The application of synchronized time stamping via the Global Positioning System (GPS) makes it feasible to measure the phasors of harmonic voltages and currents in power networks. However, due to the high expense of harmonic instruments and installation of communication channels, only a limited number of meters are currently available in power networks.

The task of harmonic state estimation (HSE) [3][4] is to locate harmonic sources and estimate harmonic voltage and current distribution with wide-area harmonic measurements. However, the limited number of harmonic measurements causes the measurement matrix to often be singular or ill-conditioned.

When the network is only partially observable, Singular Value Decomposition (SVD) [5] based method is proposed to estimate state variables in observable islands while the rest of the state variables remain unknown. In order to minimize meter requirements as well as to ensure observability, optimal meter placement is addressed in [6][7]. Reference [8] describes the application of HSE to an actual power system, where eight synchronized phasor measurements are used while state variables are seven unknown nodal harmonic current injections.

Despite these efforts, it is still a challenge to estimate reliably all network state variables in even moderate size power networks when provided fewer measurements than suspicious nodes.

Existed works seldom utilize the spatial sparsity of harmonic sources. That is the simultaneous number of large harmonic sources is small compared to suspicious node number in practical power systems. In this paper, by extending the concept of observability, we show that the estimation problem in the underdetermined system can be solved uniquely by using the sparsity of harmonic sources.

II. Problem Description

For harmonic analysis in electric transmission systems, we model harmonic sources modeled as current sources, and other equipments as constant impedance.

Given harmonic current injections $\dot{\mathbf{I}}(h)$ and harmonic nodal admittance matrix $\mathbf{Y}(h)$, nodal harmonic voltages $\dot{\mathbf{V}}(h)$ can be obtained by solving harmonic power flow equations based as follows:

$$\mathbf{Y}(h)\dot{\mathbf{V}}(h) = \dot{\mathbf{I}}(h) \tag{1}$$

where h stands for harmonic order. The branch harmonic currents $\dot{\mathbf{I}}_b(h)$ can be obtained subsequently.

Harmonic state estimation estimates network state variables with available measurements. Since harmonic source injections can determine all other network variables uniquely, $\dot{\mathbf{I}}(h)$ can be used as state variables.

We choose a subset of nodal voltages $\dot{\mathbf{V}}(h)$ and branch currents $\dot{\mathbf{I}}_b(h)$ as measurements, with all nodal current injections $\dot{\mathbf{I}}(h)$ as state variables. We assume that network topology and parameters in all considered harmonic orders are known. After splitting complex variables into real and imaginary components, the relationship between measurements and state variables can be formulated as follows:

$$\mathbf{z}(h) = \mathbf{H}(h)\mathbf{x}(h) + \mathbf{e}(h) \tag{2}$$

where

h	harmonic order,
m	number of measurements,
n	number of suspicious buses, and $m < n$,
$\mathbf{z}(h)$	$m \times 1$ measurement vector,
$\mathbf{H}(h)$	$m \times n$ measurement matrix,
$\mathbf{x}(h)$	$n \times 1$ state variable vector with excluding nonsource buses,
$\mathbf{e}(h)$	$m \times 1$ measurement error vector.

$$\mathbf{z} = \begin{bmatrix} \dot{\mathbf{z}}_R \\ \dot{\mathbf{z}}_I \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \dot{\mathbf{x}}_R \\ \dot{\mathbf{x}}_I \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} \mathbf{H}_R & -\mathbf{H}_I \\ \mathbf{H}_I & \mathbf{H}_R \end{bmatrix}$$

Branch current measurements are related to $\mathbf{x}$ by corresponding rows of nodal-branch distribution factor matrix. Nodal voltage measurements are related to $\mathbf{x}$ by corresponding rows of nodal impedance matrix. Note that nonsource buses are reduced during pre-processing steps. The measurement errors are assumed as i.i.d normal distribution with small variance.

In practical power systems, it is observed that the distribution of harmonic sources has *spatial sparsity*, that is non-neglectable harmonic sources appearing only at a small fraction of buses simultaneously. Denoting the nodal harmonic current injection vector by $\mathbf{x}$, sparsity means

$$\|\mathbf{x}\|_0 \leq s \tag{3}$$

where $\|\cdot\|_0$ is L_0 norm, which equals to the number of nonzero entries. s gives the maximum number of simultaneous harmonic sources. In practice $s \ll n$. We assume the number of measurements $m > s$ in our analysis.

Considering the spatial sparsity of harmonic sources, we formulate harmonic state estimation as a constrained approximated sparsity maximization:

$$\begin{array}{cc} \min\limits_{\mathbf{x}} & \|\mathbf{x}\|_1 \\ \text{subject to} & \|\mathbf{z} - \mathbf{H}\mathbf{x}\|_\infty \leq \varepsilon \end{array} \tag{4}$$

where L_1 norm $\|\mathbf{x}\|_1 \triangleq \sum_{k=1}^{n} |x_k|$ is used to approximate L_0 norm, L_∞ norm $\|\mathbf{v}\|_\infty \triangleq \sup_{1\leq k \leq n} |v_k|$, scalar $\varepsilon > 0$ controls the tolerance to measurement errors, and all measurements are normalized to have the same standard variance. In the following sections we will show that (4) can give an accurate estimate to the underdetermined system (2) under certain conditions.

III. Extended Observability Analysis

Observability analysis determines the necessary conditions for the uniqueness of estimates. An observable linear estimator generally requires full column rank of its measurement matrix. In this section, we will show the underdetermined linear system (2) can become observable with sparsity prior in its sources. A closely related topic in signal processing is called optimally sparse representation. This paper uses some generalized results in [10].

Definition 1: vector $\mathbf{x}$ is *s-sparse* if only s of its entries are nonzero.

Definition 2: The *spark* of matrix $\mathbf{A}$ is defined as the smallest possible number of its columns that are linearly dependent.

Definition 3: A system is *observable* if its internal state $\mathbf{x}$ can be uniquely determined by its output $\mathbf{z}$.

Definition 4: A system is *s-observable* if it is observable when its internal state $\mathbf{x}$ is at least s-sparse, i.e. $\|\mathbf{x}\|_0 \leq s$

Suppose output $\mathbf{y} = [y_1, y_2]^T$ is generated by the linear equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}_{2\times 3}\mathbf{x}^*_{3\times 1} = [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix}$$

where the internal state variable $\mathbf{x}^*$ is 1-sparse. To obtain $\mathbf{x}^*$ from $\mathbf{y}$, we design a 3-step test. At the k-th step, we check mismatch

$$r = \|\mathbf{y} - \alpha_k x_k\|_\infty \tag{5}$$

Obviously, if any two of the column vectors $\alpha_1, \alpha_2, \alpha_3$ is independent, i.e., $spark(\mathbf{A}) = 3$, only $x_2 = d$ can achieve zero mismatch. Thus, the unique solution is given by $x_1 = 0$, $x_2 = d$, $x_3 = 0$. Therefore, the underdetermined system is observable if $spark(\mathbf{A}) = 3$ and $\mathbf{x}$ is 1-sparse.

We can generalized the observation to general underdetermined linear system

$$\mathbf{y}_{m\times 1} = \mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} \tag{6}$$

where $\mathbf{y}$ denotes measurable output, $\mathbf{x}$ denotes state variables, $\mathbf{A}$ is a known matrix, and $m < n$. The conditions for its observability are given by the following theorem:

Theorem 1 (Observability with sparse prior): The underdetermined linear system (6) is observable if $\mathbf{x}$ is s-sparse and $s < \frac{1}{2} spark(\mathbf{A})$, where matrix $\mathbf{A}$ is known and $m < n$.

Proof: (Proof by Contradiction.) According to the definition of observability, (6) is observable if it has unique solution. Assume we have non-unique s-sparse solutions $\mathbf{c}$ and $\mathbf{d}$, $\mathbf{c} \neq \mathbf{d}$, such that

$$\begin{array}{c} \mathbf{y} = \mathbf{A}\mathbf{c} = \sum_{i=1}^{s} c_{k_i}\alpha_{k_i} \\ \mathbf{y} = \mathbf{A}\mathbf{d} = \sum_{j=1}^{s} d_{p_j}\alpha_{p_j} \end{array} \tag{7}$$

where α_{k_i} is the k_i-th column of $\mathbf{A}$. Easily to see

$$\sum_{i=1}^{s} c_{k_i}\alpha_{k_i} + \sum_{j=1}^{s}(-d_{p_j})\alpha_{p_j} = \mathbf{0} \tag{8}$$

which has at most $2s$ different column vectors. Because $spark(\mathbf{A}) > 2s$, any $2s$ or less than $2s$ column vectors of $\mathbf{A}$ must be linear independent. Therefore (8) can never be true.

An exhaustive search algorithm can determine the unique solution. It is to test all of possible combinations of s non-zero entries of $\mathbf{x}$. Among all combinations $[x_{k_1}, \dots, x_{k_s}]$, only the one corresponding to the unique solution $\mathbf{x}^*$ can satisfy the equation

$$\sum_{i=1}^{s} x_{k_i}\alpha_{k_i} = \mathbf{y} \tag{9}$$

It is guaranteed by the uniqueness of estimate. Since the combination number $\binom{n}{s}$ is finite, we always can find the real solution within finite steps. This completes the proof. ∎

Corollary 1 (S-Observability): The system (6) is observable if

$$\|\mathbf{x}\|_0 \le s, \text{ and } s < \frac{1}{2} spark(\mathbf{A})$$

Remark 1: It is obvious because if all $2s$ columns are linear independent, then all $k < 2s$ columns are also linear independent.

Theorem 1 indicates that if the state vector $\mathbf{x}^*$ has at most s non-zero entries, then it is possible to use only $2s$ independent measurements to estimate $\mathbf{x}^*$ if the corresponding $\mathbf{A}$ satisfies $spark(\mathbf{A}) > 2s$. In other words, it is possible to estimate n sparse variables with m $(2s < m < n)$ measurements.

Note that among all solutions to underdetermined system (6) there is only one satisfying $\|\mathbf{x}\|_0 < \frac{1}{2} spark(\mathbf{A})$. Moreover, when sparsity prior is applied, it is also the only solution. It results in the following corollary:

Corollary 2 (The sparest solution is unique): With sparsity prior

$$\|\mathbf{x}\|_0 < \frac{1}{2} spark(\mathbf{A})$$

the sparsest solution for (6) is also the unique solution.

We apply Corollary 2 to harmonic state estimation with the sparsity condition of source distribution. The HSE problem becomes to find the sparsest solution $\hat{\mathbf{x}}$ as well as minimizing the residual $\|\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}\|_\infty$. It is formulated as

$$\begin{array}{cc} \min\limits_{\mathbf{x}} & \|\mathbf{x}\|_0 \\ \text{subject to} & \|\mathbf{z} - \mathbf{H}\mathbf{x}\|_\infty \le \varepsilon \end{array} \tag{10}$$

When measurement noises are negligible, (10) becomes a sparse representation problem[10] (P0),

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{z} = \mathbf{H}\mathbf{x} \tag{11}$$

The observability of the underdetermined state estimator is guaranteed by choosing a proper measurement matrix $\mathbf{H}$ such that

$$spark(\mathbf{H}) > 2s \tag{12}$$

where s is the possible maximum number of simultaneous major harmonic sources in the network.

Because $\|\mathbf{x}\|_0$ is not a convex function, it is difficult to obtain the global optima of (10) by standard convex programming. The problem (10) has a combinatorial nature. A naive strategy mentioned to locate the harmonic sources is to try all possible combinations of k source locations($k \le s$) and choose the sparsest one with lower-than-threshold residual. Even when s is a moderate number, the naive search algorithm has to deal with an exponential number of potential combinations, which is $\Sigma_{k=1}^{s}\binom{n}{k}$. For instance, when $s = 5$, $n = 100$, the number is around 7.9×10^7.

IV. Sparsity Maximization by L_1 Norm

To avoid the difficulties of solving the sparsity maximization problem (10), there is a series of efforts(generalized in [11] and [10]) to find a good approximation of (10) by replacing $\|\mathbf{x}\|_0$ by other functions $g(\mathbf{x})$. In particular, $g(\mathbf{x}) = \|\mathbf{x}\|_1$ is favored due to its simplicity. The corresponding constrained L_1 norm minimization problem (P1) is

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{z} = \mathbf{H}\mathbf{x} \tag{13}$$

The error between the solution of (11) and (13) is bounded. To see it, suppose $\mathbf{x}^*$ is the solution of (11), and $\hat{\mathbf{x}}$ is the solution of (13). Immediately, we have

$$\|\hat{\mathbf{x}}\|_1 \le \|\mathbf{x}^*\|_1 \tag{14}$$

The error between $\mathbf{x}^*$ and $\hat{\mathbf{x}}$ is

$$\|\mathbf{x}^* - \hat{\mathbf{x}}\|_1 \le \|\mathbf{x}^*\|_1 + \|\hat{\mathbf{x}}\|_1 \le 2\|\mathbf{x}^*\|_1$$

Since $\mathbf{x}^*$ is the real solution of (11) when (11) is s-observable , $\|\mathbf{x}^*\|_1$ is bounded. So does the estimation error.

When measurement noises appear, we use (4) to replace (13). Our numerical experiments show that the estimate from (4) is stable under small model and measurement disturbances if the observability condition in Theorem 1 is satisfied.

The optimization problem (4) can be cast into a standard convex program by applying $\mathbf{x} = \mathbf{x}_p - \mathbf{x}_n$, $\mathbf{x}_p \ge 0$, $\mathbf{x}_n \ge 0$. We have

$$\begin{array}{cc} \min\limits_{\mathbf{x}_p, \mathbf{x}_n} & f = \gamma \mathbf{1}^T(\mathbf{x}_p + \mathbf{x}_n) \\ \text{subject to} & -\varepsilon \le \mathbf{z} - \mathbf{H}(\mathbf{x}_p - \mathbf{x}_n) \le \varepsilon \\ & \mathbf{x}_p \ge 0, \mathbf{x}_n \ge 0 \end{array} \tag{15}$$

It can be solved efficiently by standard linear programming methods such as Simplex method and Interior point method. [12]

V. Numerical Experiments and Discussion

We choose IEEE 14-bus test system[9] to test the proposed method . It is a balanced system with a total of 14 buses. We assume all nodes except node 7 can have harmonic source injections. Thus there are 13 suspicious nodes. For each harmonic order, two harmonic sources are randomly placed in the network. We choose a 9-meter group in the IEEE 14-bus test system shown in Fig. 1 using a greedy search algorithm. The meter group measures the harmonic currents

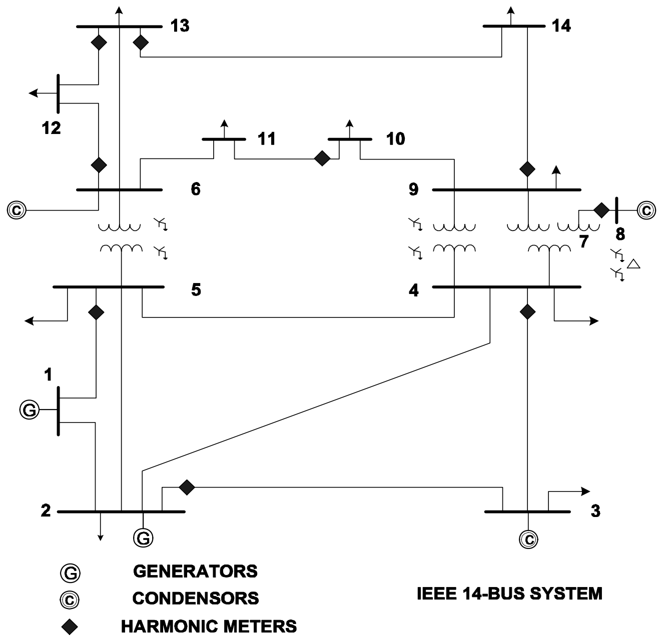

Fig. 1. IEEE 14-bus test system

through line $1-5$, $2-3$, $3-4$, $6-12$, $7-8$, $9-14$, $10-11$, $13-12$ and $13-14$. The calculated spark of the corresponding complex measurement matrix is 10. As a result of Theorem 1, the underdetermined estimator can handle up to $10/2=5$ simultaneous harmonic sources without the presence of noises.

We use the artificial harmonic injections as "actual" harmonic sources, labeled $\dot{\mathbf{I}}^{(act)}$. "Actual" nodal harmonic voltages $\dot{\mathbf{V}}^{(act)}$ are calculated by the harmonic power flow using the "actual" harmonic injections.

The measurement data are generated by harmonic power flow equations (1) with given harmonic admittance matrices and current injections. Measurement noises are added if necessary. Measurement errors are modeled as i.i.d. normal distribution with zero mean.

The harmonic state estimation is repeated for each harmonic order to obtain the injection estimate $\dot{\mathbf{I}}^{(est)}$. Then the estimated harmonic nodal voltages $\dot{\mathbf{V}}^{(est)}$ are calculated using harmonic power flow (1) and estimated current injections.

The state estimator only uses measurements and measurement matrices. Other information such as the location, magnitude and number of harmonic sources is unknown before state estimation is finished. The program is coded using Matlab 7.0 . Simplex method is used to solve the linear programming (15).

A. Experiment 1 (noiseless measurements)

In the experiment, we assume the measurement noises are zero. We set the tolerance parameter for equation (4) to 0.001. For each harmonic order, Harmonic sources can appear at any two buses except the non-source bus 7. The proposed state estimation algorithm (4) is conducted for each harmonic order. The estimated and simulated data are compared in Fig. 2 and Fig. 3, for injection magnitude and voltage magnitude respectively. We can see the estimation errors are almost zero for both voltages and injection magnitude. Moreover both the location and the magnitude of unknown harmonic sources are identified correctly and precisely.

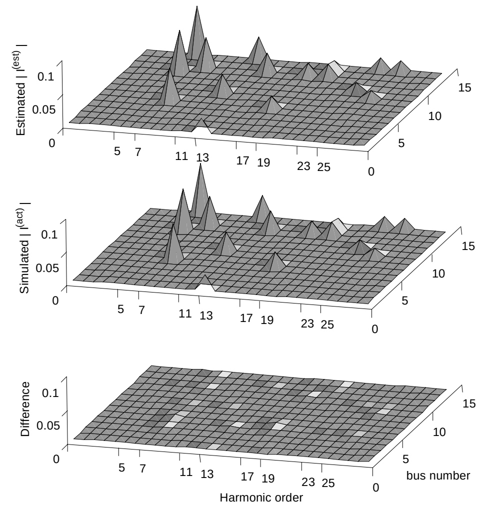

Fig. 2. Comparison of estimated and simulated nodal harmonic current injection magnitude in each harmonic order with noiseless measurements for IEEE 14-bus test system.

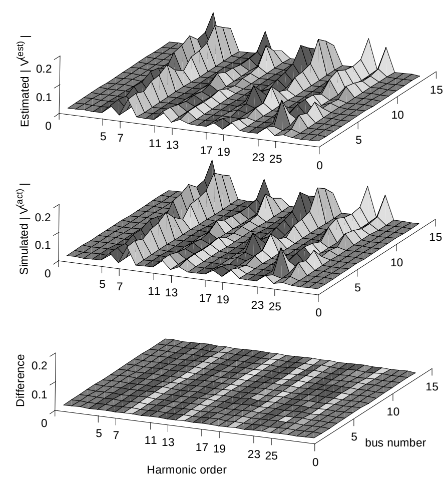

Fig. 3. Comparison of estimated and simulated nodal harmonic voltage magnitudes in each harmonic order with noiseless measurement for IEEE 14-bus test system.

B. Experiment 2 (noisy measurements)

We test to see if the proposed algorithm is stable with presence of measurement noises. We assume the measurement noises are normal distribution with 5% standard deviation and set the tolerance parameter ε as 10%. All the other settings are the same as that in experiment 1. Two harmonic sources are placed in two randomly selected buses in each harmonic order. The proposed estimation algorithm is performed to obtain

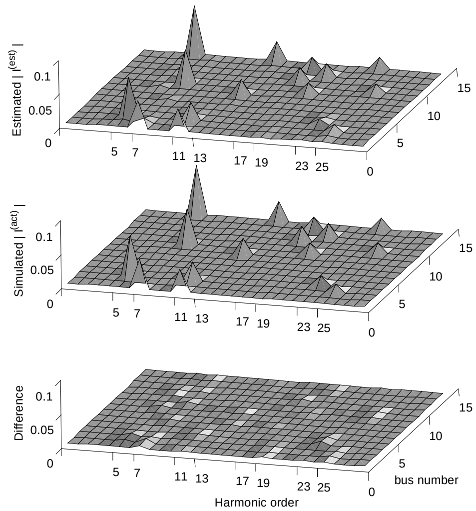

Fig. 4. Comparison of estimated and simulated nodal harmonic current injection magnitude in each harmonic order for IEEE 14-bus test system when measurement noises have a 5% standard deviation.

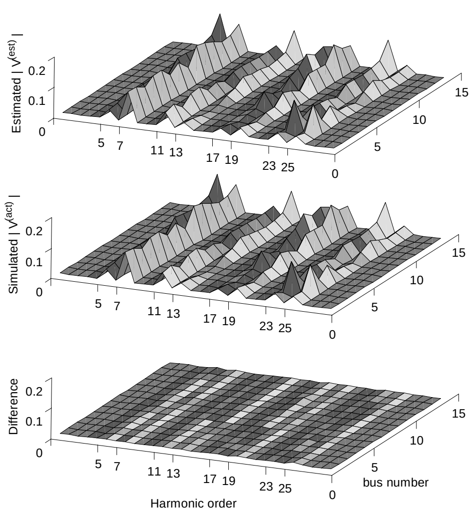

Fig. 5. Comparison of estimated and simulated nodal harmonic voltage magnitude in each harmonic order for IEEE 14-bus test system when measurement noises have a 5% standard deviation.

injection estimate for each harmonic order. Nodal voltages are calculated using the estimated injection afterwards. Estimated and simulated harmonic injection currents are compared in Fig. 4. The comparison between estimated and simulated voltage magnitude is shown in Fig. 5.

The results show that all harmonic sources in all harmonic orders are identified correctly with small magnitude differences from the simulated values. The root mean square errors (RMS) of voltage magnitude $V_M(\%)$, voltage angle V_A, injection magnitude $I_M(\%)$, and injection angle I_A in each harmonic order are listed in TABLE I, where the RMS injection errors are averaged by the number of sources and RMS voltage errors by the total node number.

TABLE I

ROOT MEAN SQUARE ERRORS BETWEEN ESTIMATED AND SIMULATED VALUES. $V_M(\%)$, V_A, $I_M(\%)$ AND I_A, ARE THE RMS ERROR OF VOLTAGE MAGNITUDE, VOLTAGE ANGLE, INJECTION MAGNITUDE AND INJECTION ANGLE, RESPECTIVELY.

	5th	7th	11th	13th	17th	19th	23rd	25th
$V_M(\%)$	1.43	1.77	0.54	2.46	1.69	1.89	2.25	3.42
$V_A(^\circ)$	0.41	0.06	0.03	0.58	2.69	1.49	0.84	0.32
$I_M(\%)$	2.17	0.42	2.15	2.83	0.92	9.21	1.84	6.60
$I_A(^\circ)$	1.87	0.57	0.01	0.13	0.49	1.72	0.56	0.50

C. Discussion

The numerical experiments show that the proposed underdetermined estimator is capable of identifying the harmonic sources reliably for both noiseless and noisy measurements. Moreover the calculated nodal voltage phasors using estimated injections have only small deviations from the simulated values. Since Theorem 1 is also valid for overdetermined systems, our numerical results (not listed here) show that the proposed algorithm is able to obtain an accurate estimate for ill-conditioned overdetermined systems while least square estimator fails and SVD estimator obtains a reliable estimate only for partial networks.

VI. CONCLUSION

The application of harmonic state estimation to power networks is hampered by the limited number of available harmonic measurements. In the underdetermined system, full observability cannot be ensured. It leads to the failure of standard least square based methods. By utilizing the spatial sparsity of harmonic sources, we extend the traditional observability analysis by showing that the underdetermined system can become observable under proper measurement arrangements. Then the estimation problem is formulated as a sparsity maximization problem which can be solved efficiently by linear programming. The proposed algorithm is tested in a balanced IEEE 14-bus test system. Our results show that we can obtain reliable harmonic estimate for the underdetermined system with 13 unknown sources and only 9 meters when small measurement noises appear. As we know so far, other methods require the number of measurement at least equals to the suspicious bus number, which is 13 in this case.

The extended observability analysis developed in this paper can also be applied to estimate configuration changes for general state estimation.

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