# Modal Analysis of MIMO Capacity in a Hallway


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*Abstract*- Systems with multiple element transmitter and receiver arrays have been shown to achieve very high spectral efficiencies. This is especially important in wireless applications that are power, bandwidth and complexity limited. The achievable capacity depends on the channel between the transmitters and the receivers, and a common assumption is that of a rich scattering environment. However an environment such as a hallway is more closely approximated as a waveguide, and we will look into the limitations this poses on the achievable rates. We will first show that the number of allowable propagation modes in a lossless waveguide limits the number of equivalent spatial sub-channels. We will then apply a similar analysis to a lossy waveguide, where the losses are due to imperfectly conducting boundaries and imperfections in the dielectric. For demonstration purposes we will show these effects for a waveguide of small dimensions. A hallway, which is a more practical environment for wireless applications, can be treated as an overmoded waveguide. In a hallway the losses are higher because the boundaries are dielectric materials that allow high power leakage. We will again demonstrate the above effects, taking into account the losses off the walls.


## I. INTRODUCTION

Assume a system with $N_t$ transmitters and $N_r$ receivers. Each transmitter i transmits an independent data stream $x_i$. The total transmitted power is $P_t$ and is equally distributed on all the transmitters. Let $E_x$ be the power from each transmitter ($P_t = N_t E_x$).

If the channel is narrowband, the channel gain between transmitter i and receiver j is a scalar $T_{ji}$. Let $y_j$ be the received signal on receiver j, and let $\underline{y}$ be the $N_r$ -dimensional vector of all the received signals. The transmitted and the received signals are related by an equation of the form $\underline{y} = T\underline{x} + \underline{n}$, where $\underline{n}$ is the noise vector (the channel transfer matrix T is $N_r x N_t$ dimensional).

The above notation also holds for the base-band representation of the signals and the channel, so the elements of $\underline{x}$, $\underline{y}$, $\underline{n}$, T can be complex.

The noise is assumed to be additive white noise with mean zero and variance $\sigma^2$ and its components are assumed to be independent across the receivers.

The mathematical expression for the channel capacity when there is no channel feedback to the transmitters is

$$C = \log_2(\det(I + \frac{E_x}{\sigma^2} TT^H)) \quad (1)$$

We re-write the matrix T in its singular value decomposed form $T=SUV^H$. The matrices S, V are unitary and the matrix U contains the singular values of T. Then $TT^H = S\Lambda S^H$, where the matrix $\Lambda$ is diagonal, and

$$C = \log_2 \prod_{i=1}^{K} (1 + \frac{E_x}{\sigma^2} \lambda_i) = \sum_{i=1}^{K} \log_2 (1 + \frac{E_x}{\sigma^2} \lambda_i) \quad (2)$$

where the $\lambda_i$'s are the absolute values of the singular values of T squared.

The number of non-zero eigenvalues is the rank K of the matrix and $K \le \min(N_t, N_r)$. In essence the channel is equivalent to K parallel scalar sub-channels, each of gain $\lambda_i$.

The maximum achievable capacity for a given average signal to noise ratio ρ can be shown to be

$$C_{\max} = \min(N_t, N_r) \log_2 (1 + \frac{\rho}{\min(N_t, N_r)} N_r) \quad (3)$$

and is attainable when all the subchannels have equal gains.

It is commonly assumed in the literature that the wireless channel can be modeled as a rich scattering environment. Recent analysis ([2], [3]) however describes situations that indicate the potential limiting behavior of the channel. In this paper we will illustrate such a scenario.

We will use the theoretical capacity of multiple input- multiple output (MIMO) systems, and we will apply it to propagation in a waveguide. Section II contains the mathematical formulation for lossless waveguides and section III extends the analysis to lossy waveguides. Section IV applies these results to a waveguide of small dimensions for illustration purposes. Section V shows how these results change when they are applied to a hallway, if we treat it as an over-moded wave-guide. Section VI summarizes our conclusions.


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## II. THEORETICAL ANALYSIS FOR A LOSSLESS WAVEGUIDE

In a waveguide of rectangular cross-section, we define a coordinate system where the x- and y-axes are aligned with the sides of the wave-guide, the z- axis runs along the wave-guide and the origin is at the lower left corner of the rectangular cross-section. Let the dimensions of the cross-section be a, b in the x- and y-directions respectively. We distinguish two kinds of propagating modes: the transverse electric (TE) and the transverse magnetic (TM) modes, that have a zero component of the electric and the magnetic field along the z-axis respectively. The electric field for each mode is described by the following equations:

TABLE I
FIELD CHARACTERISTICS FOR TE AND TM MODES

| | TE | TM |
|---|---|---|
| $E_x$ | $A_{mn}\cos(\frac{\pi m}{a}x)\sin(\frac{\pi n}{b}y)$ | $B_{mn}\cos(\frac{\pi m}{a}x)\sin(\frac{\pi n}{b}y)$ |
| $E_y$ | $A_{mn}\frac{na}{mb}\sin(\frac{\pi m}{a}x)\cos(\frac{\pi n}{b}y)$ | $-B_{mn}\frac{mb}{na}\sin(\frac{\pi m}{a}x)\cos(\frac{\pi n}{b}y)$ |
| $\omega_c$ | $\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$ | $\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$ |
| $k_z$ | $\sqrt{\left(\frac{2\pi}{\lambda}\right)^2-\left(\frac{m}{a}\right)^2-\left(\frac{n}{b}\right)^2}$ | $\sqrt{\left(\frac{2\pi}{\lambda}\right)^2-\left(\frac{m}{a}\right)^2-\left(\frac{n}{b}\right)^2}$ |
| $\alpha_z$ | $\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2-\left(\frac{2\pi}{\lambda}\right)^2}$ | $\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2-\left(\frac{2\pi}{\lambda}\right)^2}$ |

Both TE and TM modes have the same x- and y-dependence.

For given waveguide dimensions, a mode (m,n) propagates without losses if the frequency ω is above the cut-off frequency $\omega_c$, with a propagation constant in the z direction $k_z$. If the frequency is less than the cut-off, then the mode is attenuated at a rate $\alpha_z$.

Let us order the modes in terms of their cut-off frequencies. So from now on a mode index i will correspond to a TE or TM mode (m(i), n(i)) such that $\omega_c(i)\leq\omega_c(j)$, $i\leq j$, and T(i)= TM or TE.

For a given frequency in this waveguide, let L be the index of the highest-order allowable propagating mode.

Let a source be placed sufficiently far from the plane z=0, so that at that plane the evanescent fields have disappeared. Assume that the source is excited by a sinusoid at frequency ω and let $\underline{A}$ be the vector of the resulting mode coefficients at z=0 ($\underline{A}=[A_1 \quad A_2 \quad \ldots \quad A_L]^T$).

The coefficients $A_i$ can be complex numbers to account for the different phases, and they depend on the position of the source, and the distance between the source and the plane z=0. In essence the vector $\underline{A}$ is equivalent to the channel transfer function. Under the narrowband assumption if the source were transmitting a signal of the form g(t), the resulting mode coefficients at the plane z=0 would be $\underline{A}g(t)$ .

At any point (x, y, z) with z≥0, the electric field is given by the following equations

$$E_x(x,y,z)=\sum_i A_i\cos(\frac{\pi m(i)}{a}x)\sin(\frac{\pi n(i)}{b}y)e^{jk_z(i)z}$$

$$E_y(x,y,z)=\sum_i \delta_i A_i\sin(\frac{\pi m(i)}{a}x)\sin(\frac{\pi n(i)}{b}y)e^{jk_z(i)z}$$

The factor δ is in accordance to the type of the mode according to Table 1. A similar expression can be derived for the z component of the TM modes, but, without loss of generality, we will concentrate on the x and y components.

The electric field is a sum of exponential terms. These will for some values of z add in phase and for others out of phase, therefore the electric field wil be periodic in z for x, y constant.

Each component of the electric field can be written as the inner product of the vector $\underline{A}$ and a position-dependent vector $\underline{p}$. So

$E_x=\underline{p}_x^T\underline{A}, E_y=\underline{p}_y^T\underline{A}$, where

$$\underline{p}_x=\begin{bmatrix}\cos\left(\frac{\pi m(1)}{a}x\right)\sin\left(\frac{\pi n(1)}{b}y\right)e^{jk_z(1)z}\\ \ldots \\ \cos\left(\frac{\pi m(L)}{a}x\right)\sin\left(\frac{\pi n(L)}{b}y\right)e^{jk_z(L)z}\end{bmatrix}$$

$$\underline{p}_y=\begin{bmatrix}\delta_1\sin\left(\frac{\pi m(1)}{a}x\right)\sin\left(\frac{\pi n(1)}{b}y\right)e^{jk_z(1)z}\\ \ldots \\ \delta_L\sin\left(\frac{\pi m(L)}{a}x\right)\sin\left(\frac{\pi n(L)}{b}y\right)e^{jk_z(L)z}\end{bmatrix}$$

We can further separate the z dependence by writing

$$\underline{p}_x=\underline{\underline{Z}}\underline{v}_x, \underline{p}_y=\underline{\underline{Z}}\underline{v}_y, \underline{\underline{Z}}=diag(e^{jk_z(i)z}),$$

$$(\underline{v}_x)_i=\cos\left(\frac{\pi m(i)}{a}x\right)\sin\left(\frac{\pi n(i)}{b}y\right),$$

$$(\underline{v}_y)_i=\delta_i\sin\left(\frac{\pi m(i)}{a}x\right)\sin\left(\frac{\pi n(i)}{b}y\right)$$

If we assume $N_t$ sources, each one of them gives rise to a vector of the form of the vector $\underline{A}$ at the plane z=0. Let $\underline{\underline{A}}$ be a matrix the columns of which are those excitation vectors $\underline{A}$ (this is an $LxN_t$ dimensional matrix).

Now assume that $N_r$ receivers are placed at different (x,y,z) locations, and that each one of them can pick up either the x- or the y-coefficient of the electric field (this is equivalent to assuming that small electric dipoles are placed at these locations). Let $\underline{\underline{P}},\underline{\underline{V}}$ be the matrices the columns of which are respectively the position vectors $\underline{p},\underline{v}$ defined at the receiver locations. Whether $\underline{p_x},(\underline{v_x})$ or $\underline{p_y},(\underline{v_y})$ is selected depends on whether the receiver picks up the x- or the y-component of the electric field. Then the channel transfer matrix from the $N_t$ transmitters to the $N_r$ receivers is the matrix is

$$\underline{\underline{T}} = \underline{\underline{P}}^T \underline{\underline{A}}$$

So the channel capacity can be written as

$$C = \log_2(\det(I + \frac{P_t}{N_t\sigma^2} \underline{\underline{P}}^T \underline{\underline{A}}\underline{\underline{A}}^H \underline{\underline{P}}^*)) \quad (4)$$

If moreover all the receivers are on the same z plane ($z=z_R$), we can write

$$\underline{\underline{T}} = \underline{\underline{P}}^T \underline{\underline{A}} = \underline{\underline{V}}^T \underline{\underline{Z}}^T \underline{\underline{A}} \quad (5)$$

$$C = \log_2(\det(I + \frac{P_t}{N_t\sigma^2} \underline{\underline{V}}^T \underline{\underline{Z}}^T \underline{\underline{A}}\underline{\underline{A}}^H \underline{\underline{Z}}^* \underline{\underline{V}}^*)) \quad (6)$$

For the same reasons that the electric field is periodic with z, the capacity is expected to be periodic with z as well.

The rank K of the matrix T is now bounded by $K \le \min(N_t, N_r, L)$ (not $K \le \min(N_t, N_r)$ as in section I). So L now limits the number of finite amplitude parallel spatial channel dimensions that can be created to communicate over the channel.

## III. THEORETICAL ANALYSIS FOR A LOSSY WAVEGUIDE

There are two fundamental sources of loss in a waveguide: imperfectly conducting boundaries and lossy dielectric.

In the case where the conducting boundaries are imperfect, an exact solution would require solution of Maxwell's equations in both the dielectric and conducting regions. Because this procedure is impractical for most geometrical configurations, we take advantage of the fact that most practical conductors are good enough to cause only a slight modification of the ideal solution. We therefore assume that power decays exponentially with distance and the attenuation constant is the ratio of the power loss per unit length to the average power transfer.

TABLE II
ATTENUATION FOR TE AND TM MODES

| | TE | TM |
|---|---|---|
| Z | $\eta\left[1-(f_c/f)^2\right]^{-1/2}$ | $\eta\left[1-(f_c/f)^2\right]^{1/2}$ |
| $\alpha_c$(nepers/m) | $\dfrac{R_s\oint\left[\lvert H_z\rvert^2+\lvert H_t\rvert^2\right]dl}{2Z_{TE}\oint\lvert H_t\rvert^2 dS}$ | $\dfrac{R_s\oint\left(\frac{\partial E_z}{\partial n}\right)^2 dl}{2k_c^2 Z_{TM}\int_{cs} E_z^2 dS}$ |

Similarly we can define the power loss due to imperfect dielectric.

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}} \quad (7)$$

It is interesting to note that the form of the attenuation produced by an imperfect dielectric is the same for all modes and all shapes of guides, although the amount of attenuation is a function of the cutoff frequency, which does depend on the guide and the mode.

The total power loss is the sum of the above contributions. Let $\alpha = \alpha_c + \alpha_d$. The losses can be incorporated in the formulation of the previous section in the matrix $\underline{\underline{Z}}$ that becomes $\underline{\underline{Z}}_L = diag(e^{(-\alpha(i)+jk_z(i))z})$ (8).

It is useful to observe that the higher order modes have higher cut-off frequencies and higher attenuation coefficients. This indicates that they decay faster than lower order modes and the effective rank of the matrix $\underline{\underline{Z}}_L$ is less than L. The number of equivalent parallel sub-channels diminishes with distance and so does the channel capacity.

For conventional waveguide materials, the losses are small and the degradation in average received power and the consequent capacity roll-off only becomes noticeable at high distances.

An equivalent treatment of the problem that would better describe high losses is assuming a certain (field) reflection coefficient R<1 off the walls of the waveguide. Each propagating mode corresponds to a certain incidence angle on the boundaries, i.e. a certain number of reflections per unit length. So for a mode (m, n)

$$k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}, k_z = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - k_x^2 - k_y^2}$$

$$\tan\theta_x = \frac{k_z}{k_x}, \tan\theta_y = \frac{k_z}{k_y}$$

$$\alpha = \ln(\frac{1}{R})\left(\frac{1}{a\tan\theta_x} + \frac{1}{b\tan\theta_y}\right) \quad (9)$$

## IV. DEMONSTRATION OF THE CAPACITY ANALYSIS FOR SMALL WAVEGUIDES

Our purpose is to illustrate the capacity periodicity with z, its dependence on the mode excitation, and the receiver location. For simplicity we normalize all distances to the wavelength, and we look at square systems (same number of transmitters and receivers).

Assume a rectangular waveguide of dimensions a=0.7λ, b=1.4λ. The allowable modes are:

TE: (0,1), (0,2), (1,0), (1,1)

TM: (1,1)

We will concentrate on a system with two elements on each side, and we will observe that capacity depends on distance, the locations of the receivers and the kind of excitation. Both receivers are assumed to be on the same z-plane.

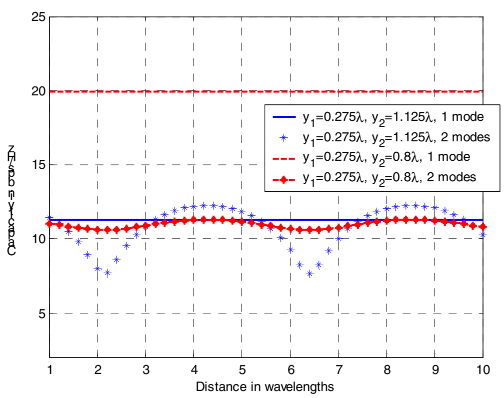


Fig. 1. Capacity of a 2x2 system for different excitations and receiver locations

Fig. 1 shows the results for two different kinds of excitation. In the first case, both transmitters excite only the (0,1) mode with the same power, and in the second, one transmitter excites the (0,1) mode and the other one the (0,2) mode. In the first case we are forcing the rank of the matrix T to be 1.

We also look at two sets of receiver locations. The first one again limits the effective rank of T, by using two locations that have the same y-dependence. Since we are looking at (0,n) modes, the x-location of the receivers is not significant.

We observe that the channel capacity is indeed periodic with z, with the same period for both excitations and receiver locations (for the case of a single excited mode, the capacity is still periodic, but the variation is too small to show in the scale of the above axes). The range within which the capacity varies is different because the spread of the eigenvalues of the matrix $\underline{\underline{T}}$ is different.

Exciting a single mode results in poor capacity performance and leads to higher capacity variation with distance. Placing the receivers at locations that have the same y-dependence also limits the rank of the transfer matrix, and therefore the channel capacity.

The effects of excitation and location are not separable. A system designer should take the combined effect into account when designing a MIMO system in such an environment.

## V. WAVEGUIDES OF PRACTICAL DIMENSIONS

We have demonstrated that the number of allowable propagating modes in a waveguide limits the channel capacity. A rough measure of the number of such modes is the ratio $(4ab)/\lambda^2$. If we consider the frequency band around 2GHz where current wireless systems operate, then the wavelength is of the order of 15cm. Given that most hallways are around 1.5m wide and 3m tall, there are about 1200 allowable modes (this number is nearly doubled if we consider both TE and TM). The number of allowable modes is not the true limiting factor. However in a hallway environment, the main source of loss is the loss through the walls. Commonly walls are dielectric materials and allow high penetration. Although there is still a large number of available modes, losses now are considerable and limit the number of significant propagating modes.

We will assume that the reflection coefficient is the same for all surfaces. This is not exactly true because of the different dielectric properties of the floors/ ceiling tend relative to the walls. It is a first order approximation to illustrate the effects we are looking at.

The following plot shows the capacity of a 6x6 system in a hallway of dimensions a=10λ, b=20λ, which would correspond to a 1.5mx3m hallway at 2GHz. In this case there are 1250 allowable modes, and we are assuming that each transmitter excites the modes in such a way that the excitation vectors are orthogonal. The receivers are arranged on a 2x3 orthogonal grid that uniformly spans the cross-section of the hallway. Fig. 2 presents the results for the cases where there is no loss (R=1), low loss (R=0.9) and high loss (R=0.5).

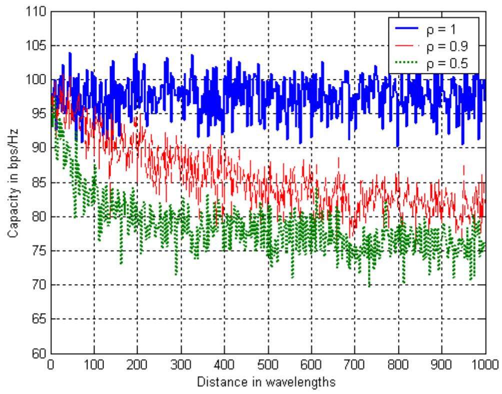

Fig. 2. Capacity dependence on distance in an overmoded lossy waveguide

We observe that the system capacity has a wilder variation with distance. This is because it is a sum of a large number of sinusoids of different frequencies (different $k_z$ 's). We also observe that the higher the loss, the faster the capacity degradation, because the average signal power falls faster. However the capacity for both lossy situations seems to stabilize at high distances, i.e. when the high-order modes are sufficiently attenuated and the low order modes dominate.

## VI. CONCLUSIONS

In this paper we have illustrated a situation where the propagation environment inherently limits the achievable capacity of a multiple input- multiple output system.

We analytically showed that in a waveguide the number of allowable propagation modes limits the number of equivalent spatial sub-channels.

Similar analysis for a lossy waveguide indicates that losses make the higher-order modes less significant, which in turn limits the achievable channel capacity. We have demonstrated how the location of the receivers and the excitation affects the system capacity performance in waveguides of small dimensions where the number of allowable propagating modes is limited.

A hallway, which is a more practical environment for wireless applications, can be treated as an overmoded waveguide. However the losses are higher because the boundaries are dielectric materials that allow high power leakage. In that case the limiting factor is not the number of allowable modes but the high attenuation that limits the number of significant modes.

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