Graph-theoretic Algorithm for Nonlinear Power Optimization Problems

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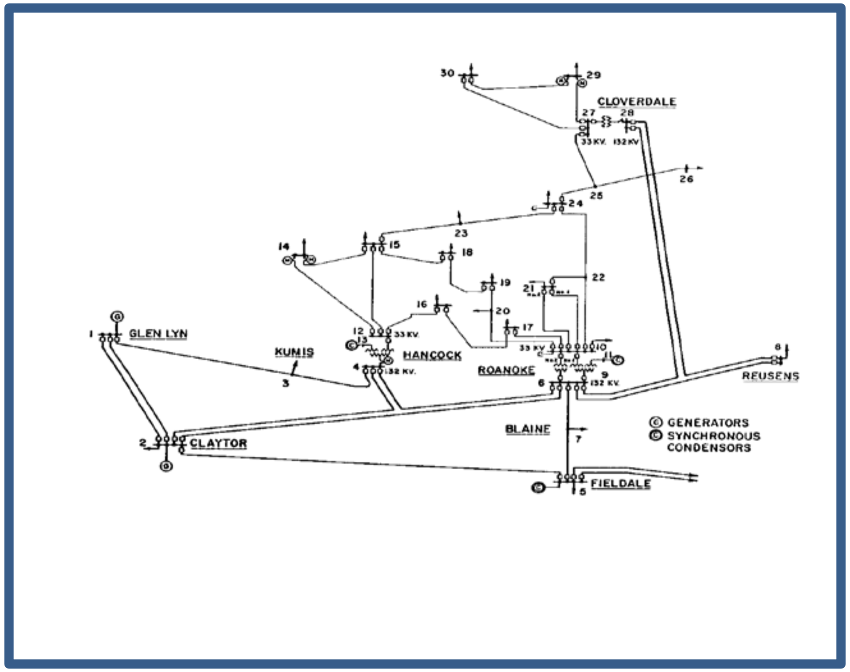
Optimization for Power Networks

- **Optimizations:**
  - Optimal power flow (OPF)
  - Security-constrained OPF
  - State estimation
  - Network reconfiguration
  - Unit commitment
  - Dynamic energy management



- **Issue of non-convexity:**
  - Discrete parameters
  - Nonlinearity in continuous variables

- **Transition from traditional grid to smart grid:**
  - More variables (10X)
  - Time constraints (100X)

Broad Interest in Power Optimization

- OPF-based problems solved on different time scales:
  - Electricity market
  - Real-time operation
  - Security assessment
  - Transmission planning

- Existing methods based on linearization or local search

- **Question:** How to find the best solution using a scalable robust algorithm?

- Huge literature since 1962 by power, OR and Econ people

Rank-Constrained Optimization

- Consider a polynomial optimization

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n-1}} \quad & f_0(x) \\
\text{s.t.} \quad & f_k(x) \leq 0 \qquad \text{for} \quad k = 1, \ldots, p
\end{aligned}
$$

- With no loss of generality, assume that f_k's are quadratic.

- Consider the transformation: $\mathbf{X} \triangleq [1 \quad x^T]^T [1 \quad x^T]$

- **Rank-constrained reformulation:**

$$
\begin{aligned}
\min_{\mathbf{X} \in \mathbb{S}^n} \quad & \text{trace}\{\mathbf{F}_0 \mathbf{X}\} \\
\text{s.t.} \quad & \text{trace}\{\mathbf{F}_k \mathbf{X}\} \leq 0 \qquad \text{for} \quad k = 1, \ldots, p \\
& X_{11} = 1 \\
& \mathbf{X} \succeq 0
\end{aligned}
$$

~~$\text{rank}\{\mathbf{X}\} = 1.$~~

→ **SDP relaxation**

Outline

- Review of four previous projects on SDP relaxation

- New results:
  - Notion of complexity for power networks (treewidth)
  - Connection between rank and treewidth
  - Case studies:
    1. **IEEE 300 bus:** Rank for security constrained OPF ≤ 7
    2. **Polish 3120 bus:** Rank for security constrained OPF ≤ 27

Projects 1-3

Project 1: How to solve a given OPF in polynomial time? (joint work with Steven Low)

- ❑ SDP relaxation works for various systems (IEEE systems, etc.).
- ❑ This is due to passivity.

Project 2: Find network topologies over which optimization is easy? (joint work with Somayeh Sojoudi, David Tse and Baosen Zhang)

- ❑ Distribution (acyclic) networks are fine (under certain assumptions).
- ❑ Transmission networks may need phase shifters.

Project 3: How to design a distributed algorithm for solving OPF? (joint work with Stephen Boyd, Eric Chu and Matt Kranning)

- ❑ A practical (infinitely) parallelizable algorithm using ADMM.
- ❑ It solves 10,000-bus OPF in 0.85 seconds on a single core machine.

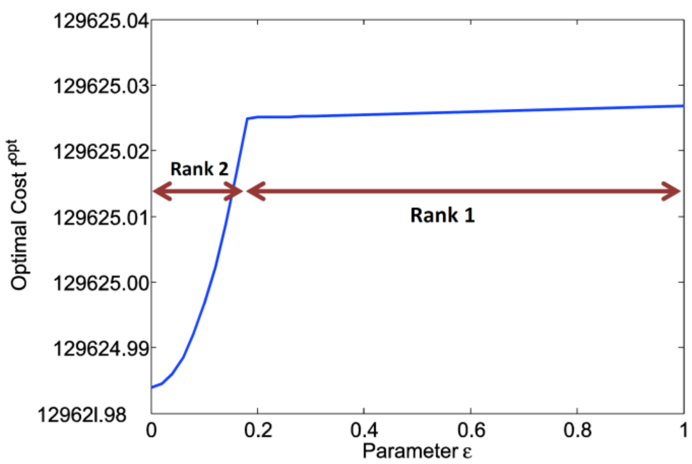
Project 4

Project 4: How to do optimization for mesh networks? (joint work with Ramtin Madani and Somayeh Sojoudi)

- ❑ **Penalized SDP relaxation:**

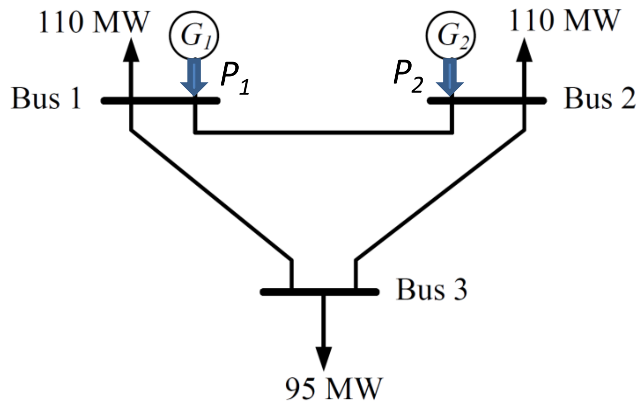
$$\sum_{k \in \mathcal{G}} f_k(P_{G_k}) \;\Longrightarrow\; \sum_{k \in \mathcal{G}} f_k(P_{G_k}) + \varepsilon \sum_{k \in \mathcal{G}} Q_{G_k}$$

- ❑ **Example borrowed from Bukhsh et al.:**
  - ❑ Modify IEEE 118-bus system has 3 local solutions with the optimal costs 129625.03, 177984.32 and 195695.54.
  - ❑ Our method finds the best one.

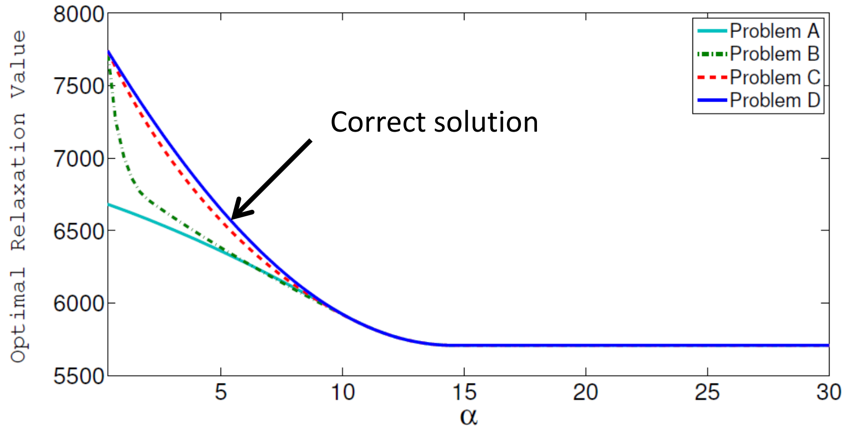


Response of SDP to Equivalent Formulations

- **Capacity constraint:** active power, apparent power, angle difference, voltage difference, current?



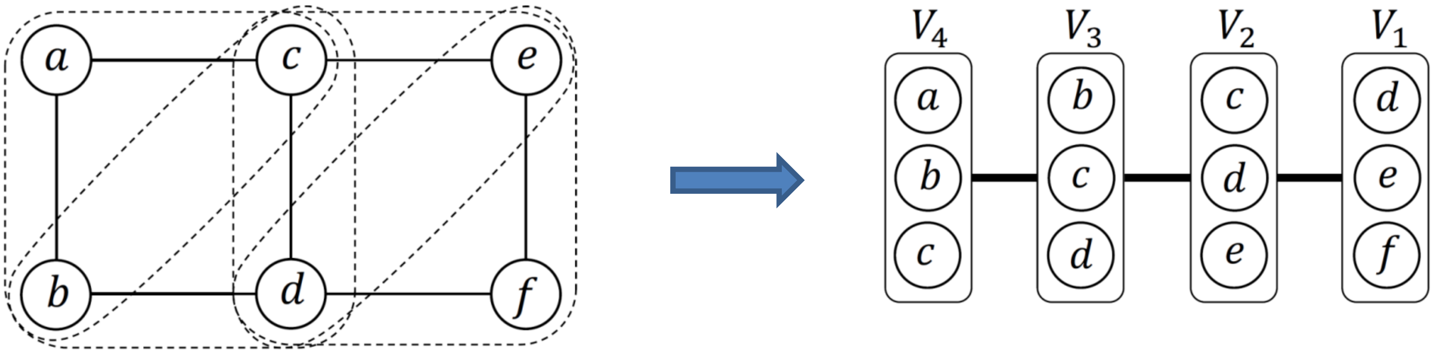
$$\text{Problem A}: \quad \theta_{13}^{\max} = \alpha$$
$$\text{Problem B}: \quad P_{13}^{\max} = \text{Re}\{(1 - e^{\alpha \mathrm{i}})y_{13}^*\}$$
$$\text{Problem C}: \quad S_{13}^{\max} = |(1 - e^{\alpha \mathrm{i}})y_{13}^*|$$
$$\text{Problem D}: \quad \Delta V_{13}^{\max} = 2\,(1 - \cos(\alpha))$$



1. Equivalent formulations behave differently after relaxation.
2. Problem D has an exact relaxation.

Treewidth

❑ **Tree decomposition:**



❑ We map a given graph *G* into a tree *T* such that:

- ❖ Each node of *T* is a collection of vertices of *G*
- ❖ Each edge of *G* appears in one node of *T*
- ❖ If a vertex shows up in multiple nodes of *T*, those nodes should form a subtree

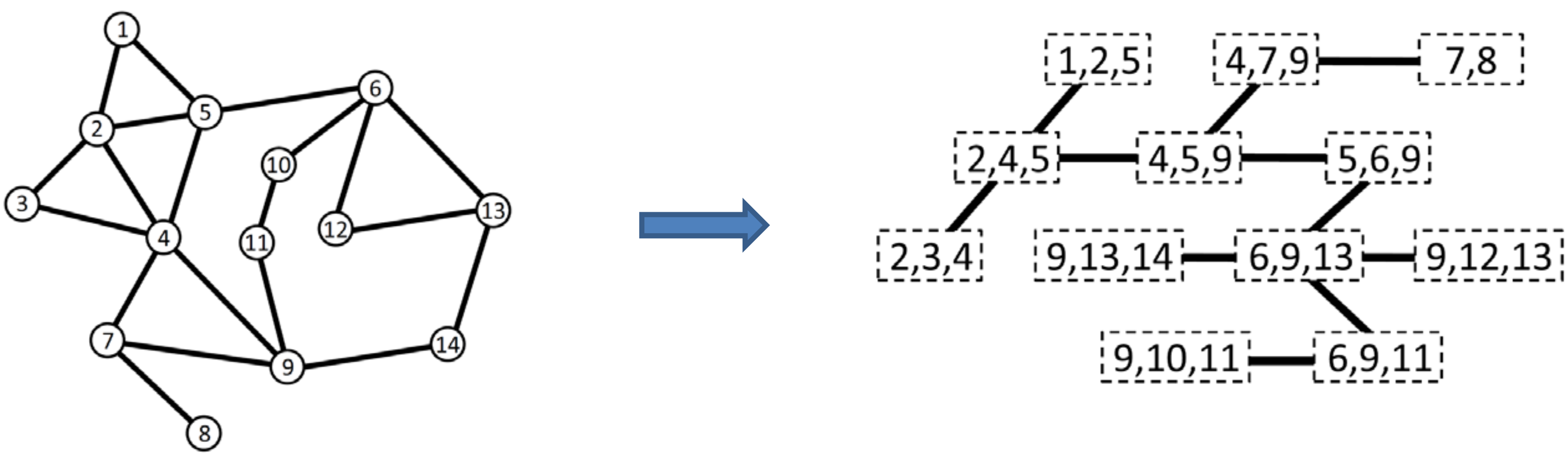
❑ **Width of a tree decomposition:** The cardinality of largest node minus one

❑ **Treewidth of graph:** The smallest width of all tree decompositions

Power Networks

- ❑ Treewidth of a tree: 1

- ❑ How about the treewidth of IEEE 14-bus system with multiple cycles? 2



- ❑ How to compute the treewidth of a large graph?
  - ❖ NP-hard problem
  - ❖ We used graph reduction techniques for sparse power networks

Power Networks

- Upper bound on the treewidth of sample power networks:

System $\mathcal{G}$	tw$\{\mathcal{G}\}$	System $\mathcal{G}$	Bound on tw$\{\mathcal{G}\}$
IEEE 14-bus	2	Polish 2383-bus	26
IEEE 30-bus	3	Polish 2736-bus	38
New England 39-bus	3	Polish 2746-bus	40
IEEE 57-bus	5	Polish 3012-bus	28
IEEE 118-bus	4	Polish 3120-bus	26
IEEE 300-bus	6	Polish 3375-bus	28

$$\begin{aligned} \min \quad & \text{trace}\{\mathbf{F}_0\mathbf{X}\} \\ \text{s.t.} \quad & \text{trace}\{\mathbf{F}_k\mathbf{X}\} \leq 0 \quad \text{for} \quad k = 1, \ldots, p \\ & X_{11} = 1 \\ & \mathbf{X} \succeq 0 \end{aligned}$$

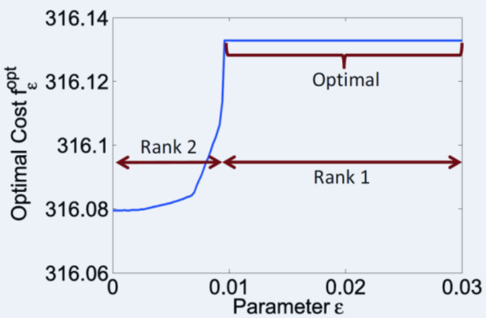
Real/complex optimization →

- Define G as the sparsity graph
- **Theorem:** There exists a solution with rank at most treewidth of *G* +1
- We proposed infinitely many optimizations to find that solution.

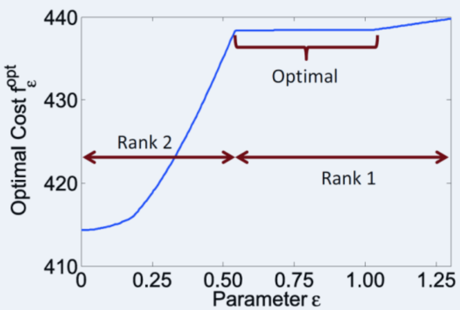
Examples

- **Example:** Consider the security-constrained unit-commitment OPF problem.
- Use SDP relaxation for this mixed-integer nonlinear program.
- What is the rank of $\boldsymbol{X}^{opt}$?
  1. **IEEE 300-bus system:** rank ≤ 7
  2. **Polish 3120-bus system:** Rank ≤ 27

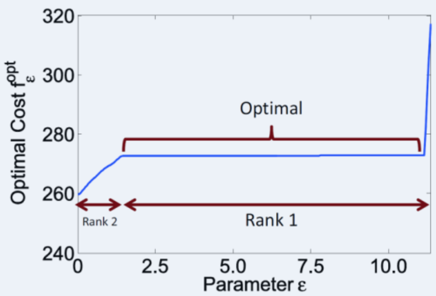
- How to go from low-rank to rank-1? Penalization (tested on 7000 examples)



IEEE 14-bus system



IEEE 30-bus system



IEEE 57-bus system

Polynomial Optimization

polynomial optimization $\Longleftrightarrow$ dense QCQP $\Longleftrightarrow$ sparse QCQP

- **Sparsification Technique:** distributed computation

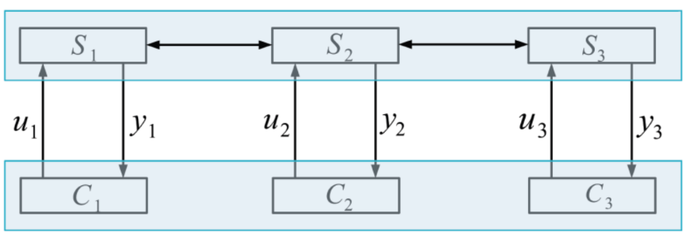
$$x_i \quad \Longleftrightarrow \quad (x_{i1}, x_{i2}) \quad \text{s.t.} \quad x_{i1} = x_{i2}$$

- This gives rise to a sparse QCQP with a sparse graph.
- The treewidth can be reduced to 2.

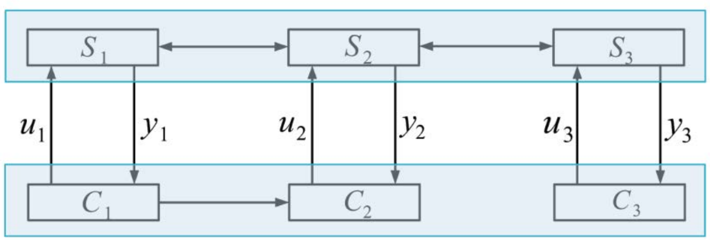
Theorem: Every polynomial optimization has a QCQP formulation whose SDP relaxation has a solution with rank 1, 2 or 3.

Distributed Control

- ❑ Computational challenges arising in the control of real-world systems:
  - Communication networks
  - Electrical power systems
  - Aerospace systems
  - Large-space flexible structures
  - Traffic systems
  - Wireless sensor networks
  - Various multi-agent systems



Decentralized control



Distributed control

Optimal Decentralized Control Problem

- **Optimal centralized control:** Easy (LQR, LQG, etc.)
- **Optimal distributed control (ODC):** NP-hard (Witsenhausen's example)

- Consider the time-varying system:

$$\begin{cases} x[\tau+1] = A[\tau]x[\tau] + B[\tau]u[\tau] \\ y[\tau] = C[\tau]x[\tau] \end{cases}, \qquad \forall \tau \in \mathbb{Z}_+$$

- The goal is to design a structured controller $u[\tau] = Ky[\tau]$ to minimize

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q[\tau] x[\tau] + u[\tau]^T R[\tau] u[\tau] \right) + \mu \ \text{trace}\{KK^T\}$$

Two Quadratic Formulations in Static Case

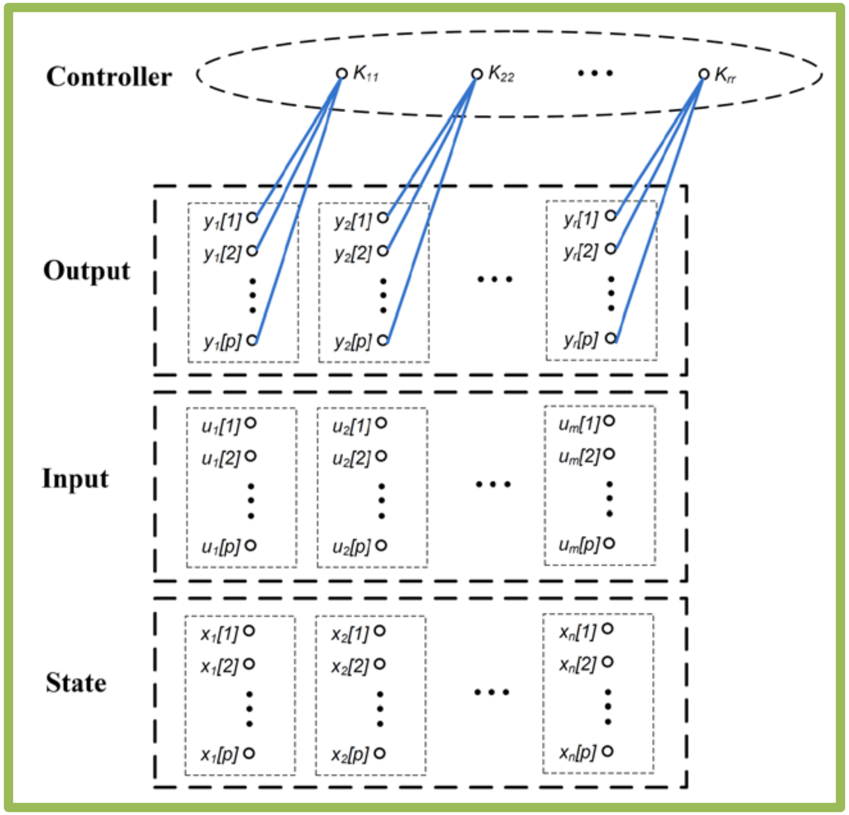
- **Formulation in time domain:**
  - Stack the free parameters of K in a vector h.
  - Define **v** as:

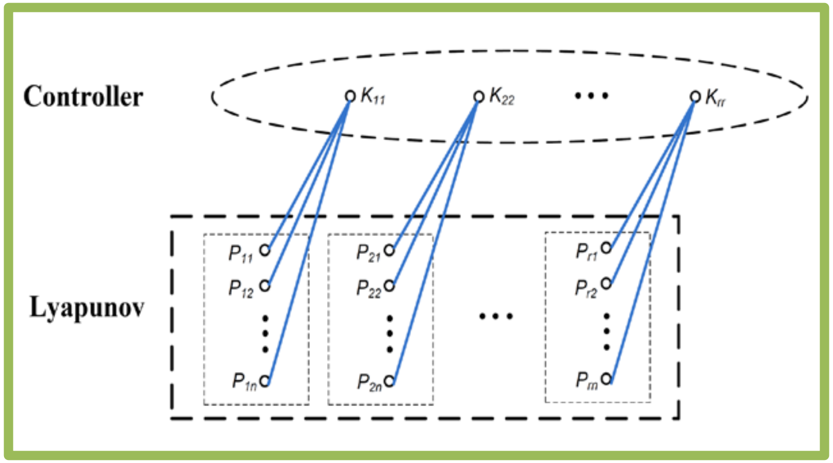
$$\mathbf{v} = \begin{bmatrix} 1 & h^* & x[0]^* & x[1]^* & \cdots & x[p]^* & y[0]^* & \cdots y[p]^* & u[0]^* & \cdots & u[p]^* \end{bmatrix}^*$$

- **Formulation in Lypunov domain:**
  - Consider the BMI constraint: $\begin{bmatrix} P & P(A+BKC)^* \\ (A+BKC)P & P \end{bmatrix} \succ 0$
  - Define **v** as:

$$v = \begin{bmatrix} 1 & h^* & P_{11} & P_{12} & \cdots & P_{1n} & \cdots & P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}^*$$

Graph of ODC for Time-Domain Formulation





Theorem: The SDP relaxation of ODC has a solution with rank 1-3 in the static/dynamic case for the time-domain or Lyapunov-domain formulation.

Conclusions

❖ **Optimization over power networks:**

- Complexity is related to treewidth.

❖ **Optimal decentralized control:**

- NP-hard problem with a small treewidth.

❖ **General theory for polynomial optimization:**

- Every polynomial optimization has an SDP relaxation with a rank 1-3 solution.

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3. G. Fazelnia, R. Madani, J. Lavaei, “Optimal Decentralized Control Problem as A Rank-Constrained Optimization,” Preprint, 2014.